Short Course on
LOGIC, ALGEBRA, AND TOPOLOGY

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Contents

1 Lecture One (May 3, 1995)

Intuitionistic propositional logic

The next definition reviews the system of intuitionistic propositional logic in terms of the sequent calculus.

Definition 1.1 Sequent System (Gentzen [1935]) *The relation $\vdash$ is inductively defined as follows:*

1. *assumptions*

   (a) $\Gamma, \beta \vdash \beta$.

2. *conjunction*

   - $\dfrac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta.}$
   - $\dfrac{\Gamma, \alpha \vdash \gamma}{\Gamma, \alpha \wedge \beta \vdash \gamma}$ *and* $\dfrac{\Gamma, \beta \vdash \gamma}{\Gamma, \alpha \wedge \beta \vdash \gamma.}$

3. *disjunction*

   - $\dfrac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta}$ *and* $\dfrac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta.}$
   - $\dfrac{\Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma, \alpha \vee \beta \vdash \gamma.}$

4. *implication*

   - $\dfrac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta.}$
   - $\dfrac{\Gamma \vdash \alpha \quad \Gamma, \beta \vdash \gamma}{\Gamma, \alpha \to \beta \vdash \gamma.}$

5. *falsity*

   - $\dfrac{\Gamma \vdash \bot}{\Gamma \vdash \alpha.}$

Heyting algebras

The logical rules determine a notion of algebraic structure that models the logic, in the following way:

1. the consequence relation $\vdash$ is mirrored by a partial ordering $\sqsubseteq$, since axioms provide reflexivity, the possibility of merging proofs provides transitivity, and the fact that if two formulas are each derivable from the other then they are equivalent provides antysimmetry;

2. conjunction $\wedge$ is mirrored by the g.l.b. operation $\sqcap$, since the second group of rules tells that conjunction is a lower bound, and the first rule tells that it is the least one;

3. disjunction $\vee$ is mirrored by the l.u.b. operation $\sqcup$, similarly;

4. implication $\rightarrow$ is mirrored by an operation $\Rightarrow$ that behaves on the right of $\sqsubseteq$ as $\sqcap$ behaves on the left, because the two rules of implication, together with the rules for conjunction, imply that

$$(\alpha \wedge \beta) \vdash \gamma \text{ iff } \alpha \vdash (\beta \rightarrow \gamma);$$

5. falsity $\perp$ is mirrored by the least element 0, since the rule tells that it implies any formula.

6. theorems are mirrored by the greatest element 1, since they are implied by any formula, being provable from no premise.

All the algebraic concepts needed to model intuitionistic propositional logic are thus standard, except perhaps the one relative to implication. This is captured by the following notion: given two functions f and g on a set partially ordered by $\sqsubseteq$, g is called a **right adjoint** of f if, for every x and y,

$$f(x) \sqsubseteq y \text{ iff } x \sqsubseteq g(y).$$

It is easy to prove that if f has a right adjoint, this is unique. Thus the following definition is not ambiguous.

Definition 1.2 *A* **Heyting algebra** *is a structure*

$$\mathcal{A} = \langle A, \sqsubseteq, \sqcap, \sqcup, \Rightarrow, 0, 1\rangle$$

such that

1. $\sqsubseteq$ *is a partial ordering*

2. $\sqcap$ *is the g.l.b. operation associated to* $\sqsubseteq$

3. $\sqcup$ *is the l.u.b. operation associated to* $\sqsubseteq$

4. $\Rightarrow$ *is the right adjoint of* $\sqcap$ *w.r.t.* $\sqsubseteq$

5. 0 *is the least element of* A *w.r.t.* $\sqsubseteq$

6. 1 *is the greatest element of* A *w.r.t.* $\sqsubseteq$.

Explicitly, the adjointness condition for $\Rightarrow$ means that

$$(a \sqcap x) \sqsubseteq y \text{ iff } a \sqsubseteq (x \Rightarrow y),$$

and examples of Heyting algebras are the following:

- **Boolean algebras**

  Let $x \Rightarrow y$ be $\overline{x} \sqcup y$, where $\overline{x}$ is the complement of x.

- **linear orderings**

  Let $x \Rightarrow y$ be 1 if $x \sqsubseteq y$, and to y otherwise.

- **open sets of topologies**

  Let $x \Rightarrow y$ be $(\overline{x} \cup y)^\circ$, where $\overline{x}$ is the complement of x, and z° the interior of z.

- **complete and $\sqcap\bigsqcup$-distributive lattices**

  Let $x \Rightarrow y$ be $\bigsqcup\{z : (z \sqcap x) \sqsubseteq y\}$.

Conversely, in any Heyting algebra $\sqcap$ preserves all existing l.u.b.'s: in particular, *any Heyting algebra is distributive*, and *a complete lattice is a Heyting algebra if and only if it is* $\sqcap\bigsqcup$*-distributive.*

Finite Heyting algebras

For a topological characterization of finite Heyting algebras, it is convenient to introduce the notion of co-point. On the one hand, this is a strengthening of the notion of atom for Boolean algebras (i.e. of an element different from 0 and with no elements between it and 0), which does not play any significant role for Heyting algebras.[1] On the other hand, this is a paradigm for further generalizations that will play a fundamental role in the following (see 1.7 and 1.10).

Definition 1.3 *An element $a \neq 0$ of a lattice is a* **co-point** *if, for every x and y,*

$$a \sqsubseteq x \sqcup y \implies (a \sqsubseteq x) \vee (a \sqsubseteq y).$$

The set of co-points of a lattice A is indicated by $\mathbf{Cpt}(\boldsymbol{A})$.

The following examples should clarify the notion of co-point:

- in a linear ordering, every element $\neq 0$ is a co-point;
- in a finite lattice, an element $\neq 0$ is a co-point if and only if it has only one immediate predecessor;
- in a distributive lattice, an element $\neq 0$ is a co-point if and only if it is $\sqcup$-irreducible, i.e.

$$a = x \sqcup y \implies (a = x) \vee (a = y);$$

[1] A Heyting algebra can be atomic (i.e. all elements different from 0 bound an atom) in a trivial way. For example, in a linear ordering there can be at most one atom, and when there is, all elements different from 0 bound the same atom: thus the atoms below the elements do not distinguish them.

- in a power set, an element is a co-point if and only if it is a singleton, i.e. of the form $\{x\}$.

Definition 1.4 *A lattice A has* **enough co-points** *if, for every x and y in A,*

$$x \neq y \implies cp(x) \neq cp(y),$$

where

$$cp(x) = \{a : a \text{ co-point } \wedge\ a \sqsubseteq x\}.$$

In other words, if $x \neq y$ then there is a co-point below one of x and y but not the other.

Equivalently, a lattice has enough co-points if every element is the l.u.b. of the co-points below it. Moreover, the following have enough co-points:

- every finite distributive lattice;
- every linear ordering;
- every power set.

We are not directly interested in complete lattices with enough co-points, since they are not automatically Heyting algebras (by the dual of 1.12 they are, up to isomorphism, exactly the algebras of *closed* sets of topologies).

Co-points however are a useful tool in the next proof, which provides a paradigm for the proofs of some crucial later results (1.9 and 1.12). On its turn, the proof is an extension of an analogous proof for Boolean algebras (with Heyting algebras, topologies and co-points replaced by, respectively, Boolean algebras, power sets and atoms).

Theorem 1.5 Topological Characterization of Finite Heyting Algebras. *The finite Heyting algebras are, up to isomorphism, exactly the algebras of open sets of finite topologies (i.e. of topologies with finitely many open sets).*

Proof. To show sufficiency, it is enough to note that the algebra of open sets of a topology is a Heyting algebra (see p. 4).

To show necessity, let A be any finite Heyting algebra, and consider the function cp from A to $\mathcal{P}(\mathrm{Cpt}(A))$ defined as follows:

$$cp(x) = \{a : a \text{ co-point } \wedge\ a \sqsubseteq x\}.$$

Also, consider on $\mathrm{Cpt}(A)$ the topology generated by $cp(A)$.

Then cp is automatically a isomorphism of Heyting algebras, for the following reasons:

- $cp(0) = \emptyset$
  By definition, for any co-point a one has $a \neq 0$, and so $a \not\sqsubseteq 0$.

- $cp(1) = \mathrm{Cpt}(A)$
  By definition, for any element a one has $a \sqsubseteq 1$.

- $x \sqsubseteq y$ *if and only if* $cp(x) \subseteq cp(y)$
  If $x \sqsubseteq y$ and $a \sqsubseteq x$ then $a \sqsubseteq y$, i.e. $cp(x) \subseteq cp(y)$.

  Conversely, if $cp(x) \subseteq cp(y)$ then all co-points below x are below y, but x is their l.u.b., and so $x \sqsubseteq y$.

- $cp(x \sqcap y) = cp(x) \cap cp(y)$
  For any element a,

  $$a \sqsubseteq (x \sqcap y) \iff (a \sqsubseteq x) \wedge (a \sqsubseteq y)$$

  by definition of $\sqcap$.

- $cp(x \sqcup y) = cp(x) \cup cp(y)$
  For any co-point a,

  $$a \sqsubseteq (x \sqcup y) \iff (a \sqsubseteq x) \vee (a \sqsubseteq y).$$

  Indeed, the right-to-left implication holds by definition of $\sqcup$, for any element a. For the left-to-right implication, let a be a co-point and $a \sqsubseteq x \sqcup y$; then $a \sqsubseteq x$ or $a \sqsubseteq y$ by definition of co-point.

- $cp(x \Rightarrow y) = (cp(x) \Rightarrow cp(y))$
  This is automatic from the previous properties of cp and the definition of topology on $\mathrm{Cpt}(A)$. More explicity, we want to show for any co-point a:

  $$a \sqsubseteq (x \Rightarrow y) \iff a \in (\overline{cp(x)} \cup cp(y))^\circ,$$

  i.e.

  $$a \sqcap x \sqsubseteq y \iff a \in (\overline{cp(x)} \cup cp(y))^\circ.$$

  By the properties of cp already proved,

  $$\begin{aligned} a \sqcap x \sqsubseteq y &\iff cp(a \sqcap x) \subseteq cp(y) \\ &\iff cp(a) \cap cp(x) \subseteq cp(y) \\ &\iff cp(a) \subseteq \overline{cp(x)} \cup cp(y) \\ &\iff cp(a) \subseteq (\overline{cp(x)} \cup cp(y))^\circ, \end{aligned}$$

  the last equivalence because $cp(a)$ is open.

  Since $a \in cp(a)$, this implies $a \in (\overline{cp(x)} \cup cp(y))^\circ$.

  Conversely, if $a \in (\overline{cp(x)} \cup cp(y))^\circ$ then, by the definition of topology on $\mathrm{Cpt}(A)$, there is z such that $a \in cp(z) \subseteq \overline{cp(x)} \cup cp(y)$. Then $a \sqsubseteq z$ and $cp(a) \subseteq cp(z)$, i.e. $cp(a) \subseteq (\overline{cp(x)} \cup cp(y))^\circ$.

- *if A has enough co-points then cp is one-one*
  This is just a restatement of the definition of having enough co-points.

  Notice that, as noted on p. 4, the hypothesis that A has enough co-points is indeed satisfied if A is finite.

- *if A is finite then cp is onto*
  The proofs given above actually show that cp preserves finite g.l.b.'s and l.u.b.'s. Thus $cp(A)$ is always closed under finite intersections and unions, since

  $$cp(x_1) \cap \cdots \cap cp(x_n) = cp(x_1 \sqcap \cdots \sqcap x_n)$$

  and

  $$cp(x_1) \cup \cdots \cup cp(x_n) = cp(x_1 \sqcup \cdots \sqcup x_n),$$

  and finite g.l.b.'s and l.u.b.'s exist in a Heyting algebra. Since A is finite so is $cp(A)$, and thus $cp(A)$ is actually closed under arbitrary intersections and unions; thus the topology generated by $cp(A)$ on the power set of the co-points of A is $cp(A)$ itself, and cp is obviously onto it. □

If A is a Heyting algebra with enough co-points, the set $\mathrm{Cpt}(A)$ of co-points of A and its topology generated by $cp(A)$ are respectively called the **Stone space** of A and the **Stone topology** associated to it.

The main property of co-points, namely

- if $a \sqsubseteq x \sqcup y$ then $a \sqsubseteq x$ or $a \sqsubseteq y$

says that the principal filter generated by a is prime.

If one replaces the function

$$cp(x) = \{a : a \text{ co-point } \wedge \ a \sqsubseteq x\}$$

by the function

$$f(x) = \{F : F \text{ is a prime filter containing } x\}$$

in the proof of 1.5, one still gets a homomorphism of Heyting algebras by the same proof. Moreover, the proof of the condition

if A has enough co-points then cp is one-one

shows that

if A has enough prime filters then f is one-one.

By noting that the hypothesis follows from the so-called **Heyting Prime Filter Theorem** (saying that, on a Heyting algebra, if $x \not\sqsubseteq y$ then there is a prime filter containing x but not y), one gets a proof of the following result.

Theorem 1.6 Stone Representation Theorem for Heyting Algebras (Stone [1937], McKinsey and Tarski [1946]) *Any Heyting algebra is isomorphic to a subalgebra of an algebra of open sets of a topology.*

This result is thus a generalization of the one-one embedding part of 1.5 to arbitrary Heyting algebras (not necessarily with enough co-points). More precisely: on the one hand, the role of co-points is taken by prime filters; on the other hand, the condition that there are enough co-points becomes the condition that there are enough prime filters, and is always satisfied (by the Heyting Prime Filter Theorem).

Thus one extends the notions of **Stone space** and **Stone topology** to arbitrary Heyting algebras (not necessarily with enough co-points), by considering the set $\mathcal{F}_A^p$ of prime filters of A and its topology generated by $f(A)$.

Topologies closed under arbitrary intersection

The proof of 1.5 shows that the collection $cp(A)$ of subsets of $\mathrm{Cpt}(A)$ (the set of co-points of A) is closed under finite unions and intersections: all $cp(A)$ lacks to be a topology, is closure under arbitrary unions. When A is finite this closure is automatic, since there are only finitely many subsets of $\mathrm{Cpt}(A)$ anyway. When A is infinite things are more complicated, but $cp(A)$ would still be closed under arbitrary unions (by the same proof) if arbitrary $\bigsqcup$ existed on the one hand, and were preserved by cp on the other.

The first condition is easy to achieve, by requiring A to be complete. For the second condition, one looks at the proof that cp preserves finite $\sqcup$, and notices that it used the notion of co-point, i.e. $\sqcup$-irreducibility; to achieve the second condition, one thus has to consider an infinitary version of the notion of co-point, i.e. $\bigsqcup$-irreducibility.

Definition 1.7 *An element $a \neq 0$ of a lattice is a* **strong co-point** *if, for every subset X,*

$$a \sqsubseteq \bigsqcup X \implies (\exists x \in X)(a \sqsubseteq x).$$

The set of strong co-points of a lattice A is indicated by **Scpt (A)***.*

The following examples, similar to those on p. 4, should clarify the notion:

- in a linear ordering, an element $\neq 0$ is a strong co-point if and only if it has an immediate predecessor;
- in a finite lattice, an element $\neq 0$ is a strong co-point if and only if it has only one immediate predecessor;
- in a $\sqcap\bigsqcup$-distributive lattice, an element $\neq 0$ is a strong co-point if and only if it is $\bigsqcup$-irreducible, i.e.

$$a = \bigsqcup X \implies (\exists x \in X)(a = x);$$

- in a power set, an element is a strong co-point if and only if it is a singleton, i.e. of the form $\{x\}$.

Definition 1.8 *A lattice A has* **enough strong co-points** *if, for every x and y in A,*

$$x \neq y \implies scp(x) \neq scp(y),$$

where

$$scp(x) = \{a : a \text{ strong co-point } \wedge \ a \sqsubseteq x\}.$$

In other words, if $x \neq y$ then there is a strong co-point below one of x and y but not the other.

Equivalently, a lattice has enough strong co-points if every element is the l.u.b. of the strong co-points below it. Moreover, the following have enough strong co-points:

- every finite distributive lattice;
- every well-ordering;
- every power set.

The next result is an analogue of 1.5.

Theorem 1.9 Topological Characterization of Complete Heyting Algebras with Enough Strong Co-points (Büchi [1952], Raney [1952]) *The complete Heyting algebras with enough strong co-points are, up to isomorphism, exactly the algebras of open sets of topologies closed under arbitrary intersections.*

Proof. To show sufficiency, first note that the open sets of a topology $\Omega(X)$ on a set X are always closed under arbitrary unions, and if they are closed under arbitrary intersections then they form a complete Heyting algebra w.r.t. the set-theoretic operations $\bigcup$ and $\bigcap$. It remains to prove the following two facts:

- *a topology closed under arbitrary intersections always has strong co-points*
  For any element $x \in X$, consider

  $$(\{x\})_\circ = \text{smallest open set containing } x,$$

  which exists because the open sets are closed under arbitrary intersections (and thus one can take the intersection of all open sets containing x). Suppose $(\{x\})_\circ \subseteq \bigcup_{i \in I} A_i$, with A_i open: since $x \in (\{x\})_\circ$, at least one of the A_i's must contain x. Since A_i is then an open set containing x, then by definition $(\{x\})_\circ \subseteq A_i$, because $(\{x\})_\circ$ is the smallest open set containing x.

- *a topology closed under arbitrary intersections always has enough strong co-points*
  Suppose A and B are distinct open sets: there must be an element x in one of them but not in the other, e.g. $x \in B - A$. Then $(\{x\})_\circ \not\subseteq A$ because $x \notin A$, and $(\{x\})_\circ \subseteq B$ because $x \in B$ and B is open. So $(\{x\})_\circ$ is a strong co-point below one of A and B but not the other.

To show necessity, let A be any complete Heyting algebra with enough strong co-points, and consider the function *scp* from A to $\mathcal{P}(\mathrm{Scpt}(A))$ defined as follows:

$$scp(x) = \{a : a \text{ strong co-point } \wedge \ a \sqsubseteq x\}.$$

Also, as in the proof of 1.5, consider on $\mathrm{Scpt}(A)$ the topology generated by $scp(A)$.

As in 1.5, *scp* is automatically a homomorphism of Heyting algebras. Moreover:

- *if A has enough strong co-points then scp is one-one*
  This is just a restatement of the definition of having enough strong co-points.

- *if A is complete then scp is onto*
  The proof of 1.5 shows that *scp* preserves arbitrary g.l.b.'s and finite l.u.b.'s. Since we are using the notion of strong co-point instead of the notion of co-point, *scp* also preserves arbitrary l.u.b.'s. If A is complete, then arbitrary g.l.b.'s and l.u.b.'s exist, and thus $scp(A)$ is closed under arbitrary unions and intersections. Thus the topology generated by $scp(A)$ on the power set of the strong co-points of A is $scp(A)$ itself, it is closed under arbitrary intersections, and *scp* is obviously onto it. □

Notice that the proof shows in particular that *a complete lattice with enough strong co-points is a Heyting algebra*, since no use of $\Rightarrow$ is made in the proof of the isomorphism.

Arbitrary topologies

The next notion is dual to 1.3.

Definition 1.10 *An element $a \neq 1$ of a lattice is a* **point** *if, for every x and y,*

$$a \sqsupseteq (x \sqcap y) \implies (a \sqsupseteq x) \vee (a \sqsupseteq y).$$

The set of points of a lattice A is indicated by $\mathbf{Pt(A)}$.

The following examples, similar to those on p. 4 and 8, should clarify the notion:

- in a linear ordering, every element $\neq 1$ is a point;

- in a finite lattice, an element $\neq 1$ is a point if and only if it has only one immediate successor;

- in a distributive lattice, an element $\neq 1$ is a point if and only if it is $\sqcap$-irreducible, i.e.
$$a = (x \sqcap y) \implies (a = x) \vee (a = y);$$

- in a power set, an element is a point if and only if it is the complement of a singleton, i.e. of the form $\overline{\{x\}}$.

Definition 1.11 *A lattice A has* **enough points** *if, for every x and y in A,*

$$x \neq y \implies p(x) \neq p(y),$$

where

$$p(x) = \{a : a \text{ point} \wedge a \not\sqsupseteq x\}.$$

In other words, if $x \neq y$ then there is a point above one of x and y but not the other.

Equivalently, a lattice has enough points if every element is the g.l.b. of the points above it. Moreover, the following have enough strong points:

- every finite distributive lattice;
- every linear ordering;
- every power set.

By the Stone Representation Theorem 1.6, every Heyting algebra is isomorphic to a *sub*algebra of a topological algebra. This leaves open the question, answered in the next result, of which Heyting algebras are actually isomorphic to the *full* topological algebra.

From a complementary perspective, every topological algebra is a Heyting algebra. This leaves open the question, also answered in the next result, of which algebraic properties of Heyting algebras are characteristic of the topological algebras.

The proof of the next result is dual to those of 1.5 and 1.9.

Theorem 1.12 Algebraic Characterization of Topologies (Papert [1959]) *The complete Heyting algebras with enough points are, up to isomorphism, exactly the algebras of open sets of topologies.*

Proof. To show sufficiency, first note that the open sets of a topology $\Omega(X)$ on a set X are closed under arbitrary unions, and hence they form a complete Heyting algebra.[2] It remains to prove the following two facts:

- *a topology always has points*
  For any element $x \in X$, consider

  $$\left(\overline{\{x\}}\right)^{\circ} = \text{interior of } \overline{\{x\}} = \text{greatest open set not containing } x,$$

  which exists because the open sets are closed under arbitrary unions (and thus one can take the union of all open sets not containing x). Suppose

[2]Notice that, while the open sets of a topology are closed under arbitrary unions and finite intersections (and hence $\bigsqcup$ and $\sqcap$ are the usual set theoretical union and intersection), they are not in general closed under arbitrary intersections (and thus the g.l.b. is only the largest open set contained in the set theoretical intersection, i.e. the latter's interior).

$\left(\overline{\{x\}}\right)^\circ \supseteq A \cap B$, with A and B open: at least one of A and B must not contain x, otherwise x would be in the intersection too. Suppose e.g. $x \notin A$: then by definition $A \subseteq \left(\overline{\{x\}}\right)^\circ$, since $\left(\overline{\{x\}}\right)^\circ$ is the greatest open set not containing x.

- *a topology always has enough points*
  Suppose A and B are distinct open sets: there must be an element x in one of them but not in the other, e.g. $x \in B - A$. Then $\left(\overline{\{x\}}\right)^\circ \supseteq A$ because $x \notin A$ and A is open, and $\left(\overline{\{x\}}\right)^\circ \not\supseteq B$ because $x \in B$. So $\left(\overline{\{x\}}\right)^\circ$ is a point above one of A and B but not the other.

To show necessity, let A be any complete Heyting algebra with enough points, and consider the function p from A to $\mathcal{P}(\mathrm{Pt}(A))$ defined as follows:

$$p(x) = \{a : a \text{ point } \wedge \ a \not\sqsupseteq x\}.$$

Also, as in the proof of 1.5, consider on $\mathrm{Pt}(A)$ the topology generated by $p(A)$.

Then p is automatically a isomorphism of Heyting algebras, for the following reasons:

- $p(0) = \emptyset$
  By definition, for any element a one has $a \sqsupseteq 0$.

- $p(1) = \mathrm{Pt}(A)$
  By definition, for any point a one has $a \neq 1$, and so $a \not\sqsupseteq 1$.

- $x \sqsubseteq y$ *if and only if* $p(x) \subseteq p(y)$
  If $x \sqsubseteq y$ and $a \not\sqsupseteq x$ then $a \not\sqsupseteq y$, i.e. $p(x) \subseteq p(y)$.

  Conversely, if $p(x) \subseteq p(y)$ then all points not above x are not above y, i.e. all points above y are above x, but y is their g.l.b., so $x \sqsubseteq y$.

- $p(x \sqcap y) = p(x) \cap p(y)$
  For any point a,

  $$a \sqsupseteq (x \sqcap y) \iff (a \sqsupseteq x) \vee (a \sqsupseteq y).$$

  Indeed, the right-to-left implication holds by definition of $\sqcap$, for any element a. For the left-to-right implication, let a be a point and $a \sqsupseteq x \sqcap y$; then $a \sqsupseteq x$ or $a \sqsupseteq y$ by definition of point.

  By taking negations,

  $$a \not\sqsupseteq (x \sqcap y) \iff (a \not\sqsupseteq x) \wedge (a \not\sqsupseteq y).$$

- $p(x \sqcup y) = p(x) \cup p(y)$
  For any element a,
  $$a \sqsupseteq (x \sqcup y) \iff (a \sqsupseteq x) \wedge (a \sqsupseteq y)$$
  by definition of $\sqcup$.

  By taking negations,
  $$a \not\sqsupseteq (x \sqcup y) \iff (a \not\sqsupseteq x) \vee (a \not\sqsupseteq y).$$

- $p(x \Rightarrow y) = (p(x) \Rightarrow p(y))$
  This is automatic from the previous properties of p and the definition of topology on $\mathrm{Pt}(A)$, as in the proof of 1.5.

- *if A has enough points then p is one-one*
  This is just a restatement of the definition of having enough points.

- *if A is complete then p is onto*
  The proofs given above actually show that p preserves finite g.l.b.'s and arbitrary l.u.b.'s. Thus $p(A)$ is always closed under finite intersections, since
  $$p(x_1) \cap \cdots \cap p(x_n) = p(x_1 \sqcap \cdots \sqcap x_n)$$
  and finite g.l.b.'s exist in a Heyting algebra. Similarly, $p(A)$ is closed under arbitrary unions when arbitrary l.u.b.'s exist on A, i.e. when A is a complete Heyting algebra. Thus if A is complete then the topology generated by $p(A)$ on the power set of the points of A is $p(A)$ itself, and p is obviously onto it.
  □

Notice that the proof shows in particular that *a complete lattice with enough points is a Heyting algebra*, since no use of $\Rightarrow$ is made in the proof of the isomorphism.

If A is a Heyting algebra with enough points, the set $\mathrm{Pt}(A)$ of points of A and its topology generated by $p(A)$ are respectively called the **dual Stone space** of A and the **dual Stone topology** (or **hull-kernel topology**) associated to it.[3]

The reason for the word 'dual' comes from the fact that we have there switched from the filters used in the previous proof of the Stone representation theorem 1.6 to ideals.[4] Indeed, the main property of points, namely

[3]A topology is homeomorphic to the dual Stone topology of a complete Heyting algebra with enough points if and only if:

1. the open sets $\left(\overline{\{x\}}\right)^\circ$ are the only points
2. if $x \neq y$ then $\left(\overline{\{x\}}\right)^\circ \neq \left(\overline{\{y\}}\right)^\circ$.

Such topologies are called **sober**.

[4]This also accounts for the slight backwardness of the definition of p, since a subset of a lattice is a prime filter if and only if its complement is a prime ideal (see also note 5 on p. 16), and thus

- if $a \sqsupseteq x \sqcap y$ then $a \sqsupseteq x$ or $a \sqsupseteq y$

says that the principal ideal generated by a is prime.

If one replaces the function

$$p(x) = \{a : a \text{ point } \wedge\ a \not\sqsupseteq x\}$$

by the function

$$i(x) = \{I : I \text{ is a prime ideal not containing } x\}$$

in the proof of 1.12, one still gets a homomorphism of Heyting algebras by the same proof. Moreover, the proof of the condition

if A has enough points then p is one-one

shows that

if A has enough prime ideals then i is one-one.

By noting that the hypothesis follows from the so-called **Heyting Prime Ideal Theorem** (saying that, on a Heyting algebra, if $x \not\sqsubseteq y$ then there is a prime ideal containing y but not x), one gets a dual proof of 1.6.

Thus one extends the notions of **dual Stone space** and **dual Stone topology** to arbitrary Heyting algebras (not necessarily with enough points), by considering the set $\mathcal{I}_A^p$ of prime ideals of A and its topology generated by $i(A)$.

While for the sake of the Stone Representation Theorem both filters and ideals eventually produce the same result, one should notice that *filters are more natural from the point of view of logic*, since they correspond to sets of formulas closed under conjunction and deduction, while *ideals are more natural from the point of view of topology*, since a topology may have no co-point at all (as the case of $\mathbb{R}$ shows), while it always has enough points (see 1.12).

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prime filters containing x correspond to prime ideals *not* containing x.

Indeed, if one defined

$$p(x) = \{a : a \text{ point } \wedge\ a \sqsupseteq x\}$$

then one would have

$$p(x \sqcap y) = p(x) \cup p(y) \qquad \text{and} \qquad p(x \sqcup y) = p(x) \cap p(y),$$

and thus p would preserve neither of $\sqcap$ and $\sqcup$.

2 Lecture Two (May 4, 1995)

Arbitrary Heyting algebras

In the previous subsection we focused on topologies, and characterized them from the point of view of Heyting algebras. In the present subsection we focus on arbitrary Heyting algebras, and characterize them from the point of view of topology.

For the characterization of arbitrary Heyting algebras in topological terms, the following notion turns out to be crucial.

Definition 2.1 *A subset X of a topological space is* **compact** *if, whenever it is covered by an union of open sets, it is already covered by a finite subunion.*

Stone topologies are actually generated by their compact open sets, as we now show.

Theorem 2.2 Topological Characterization of Heyting Algebras (Stone [1937a]) *Any Heyting algebra is isomorphic to the algebra of compact open sets of its Stone topology.*

Proof. The first proof of 1.6 shows that if $\mathcal{F}_A^p$ is the set of all prime filters on A, then the function $f : A \to \mathcal{P}(\mathcal{F}_A^p)$ defined as:

$$f(x) = \text{the set of all prime filters containing } x$$

is a one-one homomorphism of Heyting algebras.

It thus only remains to characterize $f(A)$ as the set of compact open sets.

- *every compact open set is in $f(A)$*
  Let X be a compact open set. Since X is open and $f(A)$ generates the Stone topology, there is a subset B of A such that

  $$X = \bigcup_{a \in B} f(a).$$

  Since X is compact, there is a finite subset $\{a_1, \ldots, a_n\}$ of B such that

  $$X = f(a_1) \cup \cdots \cup f(a_n).$$

  Then

  $$X = f(a_1 \sqcup \cdots \sqcup a_n)$$

  because f preserves $\sqcup$, and thus $X \in f(A)$.

- *every element of $f(A)$ is compact open*
  We first prove by contradiction that $f(1)$, i.e. the whole space $\mathcal{F}_A^p$, is compact. Suppose

  $$f(1) = \bigcup_{a \in B} f(a)$$

but, for every finite subset $\{a_1, \ldots, a_n\}$ of B,

$$f(1) \neq f(a_1) \cup \cdots \cup f(a_n).$$

Then

$$f(1) \neq f(a_1 \sqcup \cdots \sqcup a_n)$$

because f preserves $\sqcup$, and

$$1 \neq a_1 \sqcup \cdots \sqcup a_n$$

by one-onenness of f.

We want to find a prime filter F containing no $a \in B$, contradicting the fact that

$$\mathcal{F}_A^p = f(1) = \bigcup_{a \in B} f(a),$$

i.e. that every prime filter contains some a for $a \in B$.

To find F it is enough to find a prime ideal I containing every $a \in B$, and then let F be its complement (since, on any lattice, I is a prime ideal if and only if its complement is a prime filter[5]). Consider then the ideal generated by B (which consists of the downward closure of the set of all finite joins of elements of B): such an ideal is proper because, as noted above, all finite joins of elements of B are $\neq 1$.

Then the set of all proper ideals containing B is non empty and partially ordered by inclusion, and every non empty chain has a l.u.b. (which is just the union of the chain). By Zorn's Lemma there is a maximal ideal I containing B, and such an ideal is prime.

The proof that $f(x)$ is compact is a variation of the one just given for $f(1)$. Indeed, suppose

$$f(x) \subseteq \bigcup_{a \in B} f(a)$$

but, for every finite subset $\{a_1, \ldots, a_n\}$ of B,

$$f(x) \not\subseteq f(a_1) \cup \cdots \cup f(a_n).$$

[5]To show, for example, that if I is a prime ideal then $\overline{I}$ is a filter (which is what is needed above):

- *if* $x \in \overline{I}$ *and* $x \sqsubseteq y$ *then* $y \in \overline{I}$
  Suppose $y \in I$: by downward closure of I, $x \in I$.
- *if* $x, y \in \overline{I}$ *then* $x \sqcap y \in \overline{I}$
  Suppose $x \sqcap y \in I$: by primality of I, $x \in I$ or $y \in I$.
- *if* $x \sqcup y \in \overline{I}$ *then* $x \in \overline{I}$ or $y \in \overline{I}$
  Suppose $x, y \in I$: by closure under $\sqcup$ of I, $x \sqcup y \in I$.

Then

$$f(x) \not\sqsubseteq f(a_1 \sqcup \cdots \sqcup a_n)$$

because f preserves $\sqcup$, and

$$x \not\sqsubseteq a_1 \sqcup \cdots \sqcup a_n$$

because f preserves $\sqsubseteq$, in particular

$$1 \neq a_1 \sqcup \cdots \sqcup a_n .$$

As above, there is a prime filter F containing x but no element of B (since the ideal generated by B does not contain x), contradiction. □

Corollary 2.3 (Stone [1937]) *Given an arbitrary Heyting algebra, its (dual) Stone space is compact, and its (dual) Stone topology is generated by the compact open sets.*

Proof. The proof of 2.2 proves the assertion for the Stone space $\mathcal{F}^p_A = f(1)$ of a Heyting algebra A. A dual proof works for the dual Stone space. □

Algebraic Heyting algebras

In the present subsection we introduce abstract versions of the notion of compactness, and of the property of (dual) Stone topologies of being generated by (a Heyting algebra of) compact open sets.

Definition 2.4 (Birkhoff and Frink [1948], Nachbin [1949]) *Given a complete lattice A, an element a is called* **compact** *if, for every subset X,*

$$a \sqsubseteq \bigsqcup X \implies (\exists u \subseteq X, u \textit{ finite})(a \sqsubseteq \bigsqcup u).$$

The set of compact elements of a lattice A is indicated by $\mathbf{K}(\boldsymbol{A})$*.*

The following examples should clarify the notion:

- if the $\sqcap\bigsqcup$-distributive law holds, then a is compact if and only if, whenever $a = \bigsqcup X$, there is a finite subset u of X such that $a = \bigsqcup u$;
- an element of a complete lattice is a strong co-point if and only if it is a compact co-point;
- in a finite lattice, every element is compact;
- in a linear ordering an element is compact if and only if it is 0 or it has an immediate predecessor;
- in a power set an element is compact if and only if it is finite.

Definition 2.5 *A lattice A has* **enough compacts** *if, for every x and y in A,*

$$x \neq y \implies k(x) \neq k(y),$$

where

$$k(x) = \{a : a \text{ compact } \wedge\ a \sqsubseteq x\}.$$

In other words, if $x \neq y$ then there is a compact below one of x and y but not the other.

A lattice A is called **algebraic** *if it is complete and has enough compacts.*

Equivalently, a lattice has enough compacts if every element is the l.u.b. of the compacts below it. Moreover, the following are algebraic:

- every finite lattice;
- a complete linear ordering in which any two distinct elements are separated by a gap, i.e. by two elements with nothing in between;
- every power set.

It follows that not every algebraic lattice is distributive, and in particular *not every algebraic lattice is a Heyting algebra*. On the other hand, lack of distributivity is the only obstruction, since *a distributive algebraic lattice is a Heyting algebra* (by 2.12, an algebraic lattice is continuous; and as noticed after 2.14, a distributive continuous lattice is a Heyting algebra).

Theorem 2.6 Topological Characterization of Algebraic Heyting Algebras (Hofmann and Keimel [1972]) *The algebraic Heyting algebras are, up to isomorphism, exactly the algebras of open sets of topologies generated by their compact open sets.*

Proof. Sufficiency is immediate by definition 2.4, since the algebraic notion of compactness is patterned on the topological one.

To show necessity we use the fact, proved in 2.12 and 2.14, that an algebraic Heyting algebra A has enough points. By 1.12 it is then isomorphic to the algebra of open sets of the form

$$p(x) = \{a : a \text{ point } \wedge\ a \not\sqsupseteq x\}.$$

It is thus enough to notice that such a topology is generated by the compact open sets. Since the topology is generated by $p(A)$, it is enough to show that every element of $p(A)$ is the union of the compact open sets contained in it. Since every element of A is the l.u.b. of the compact elements below it, and p preserves arbitrary l.u.b.'s, it is enough to prove that the image of a compact element is compact.

Let thus a be compact on A, and $p(a) \subseteq \bigcup_{x \in X} p(x)$. Then $p(a) \subseteq p(\bigsqcup X)$ because p preserves arbitrary l.u.b.'s, and then $a \sqsubseteq \bigsqcup X$ because p is an isomorphism. But a is compact, so $a \sqsubseteq \bigsqcup u$ for some finite $u \subseteq X$, and then

$$p(a) \subseteq p(\bigsqcup u) = \bigcup_{x \in u} p(x),$$

i.e. $p(a)$ is compact. $\square$

Continuous Heyting algebras

Strong co-points and compact elements can be considered as approximations to elements, generating the lattice when there are enough of them. We now relativize compactness and introduce a further notion of approximation, which turns out to be the most appropriate for applications.

Definition 2.7 (Scott [1972]) *An element a of a lattice is* **way below** *another element x ($\boldsymbol{a \ll x}$) if a is in any ideal I such that $x \sqsubseteq \bigsqcup I$.*

The following examples should clarify the notion:

- an element a of a complete lattice is compact if and only if $a \ll a$;
- in a $\sqcap\bigsqcup$-distributive lattice, $a \ll x$ if and only if a is in any ideal I such that $x = \bigsqcup I$;
- in a finite lattice, $a \ll x$ if and only if $a \sqsubseteq x$, and $x \ll x$ for every x;
- in a linear ordering, $a \ll x$ if and only if $a \sqsubset x$ or $a = x \ll x$, and $x \ll x$ if and only if $x = 0$ or x has an immediate predecessor;
- in a power set, $a \ll x$ if and only if a is a finite subset of x, and $x \ll x$ if and only if x is finite.

The following property of $\ll$, especially the third, will be crucial in the proof of 2.14.

Proposition 2.8 *$\ll$ is stronger than $\sqsubseteq$, transitive and dense.*

Proof. To show that $\ll$ is stronger than $\sqsubseteq$, it is enough to notice that $\{a : a \sqsubseteq x\}$ is an ideal with l.u.b. x.

To show that $\ll$ is transitive, let $a \ll b \ll c$ and $c \sqsubseteq \bigsqcup I$. Then $b \in I$ because $b \ll c$; thus $a \in I$ because $a \sqsubseteq b$ (since $a \ll b$), and I is an ideal.

We now turn to the proof that $\ll$ is dense. Given $a \ll b$, consider the set

$$I = \{x : (\exists y)(x \ll y \ll b)\}.$$

It is enough to show that I is an ideal such that $b \sqsubseteq \bigsqcup I$: then $a \in I$, because $a \ll b$; and $a \ll y \ll b$ for some y, by definition of I.

- *I is downward closed*
  Suppose $z \sqsubseteq x$ and $x \in I$. Then $x \ll y \ll b$ for some y, by definition of I; and $z \ll y$ because $z \sqsubseteq x \ll y$, by definition of $\ll$. Thus $z \ll y \ll b$, and $z \in I$.

- *I is closed under $\sqcup$*
  Suppose $x_1 \ll y_1 \ll b$ and $x_2 \ll y_2 \ll b$. Since $\{z : z \ll b\}$ is an ideal (being an intersection of ideals), $y_1 \sqcup y_2 \ll b$. Since $\{z : z \ll y_1 \sqcup y_2\}$ is an ideal, $x_1 \sqcup x_2 \ll y_1 \sqcup y_2$. Thus $x_1 \sqcup x_2 \ll y_1 \sqcup y_2 \ll b$, and $x_1 \sqcup x_2 \in I$.

- $b \sqsubseteq \bigsqcup I$
  Since I is an ideal of elements $\ll b$, $\bigsqcup I \sqsubseteq b$. Suppose $\bigsqcup I \sqsubset b$: since the elements way below b have l.u.b. b, there is $y \ll b$ such that $y \not\sqsubseteq \bigsqcup I$; since the elements way below y have l.u.b. y, there is $x \ll y$ such that $x \not\sqsubseteq \bigsqcup I$. Thus $x \ll y \ll b$, i.e. $x \in I$, and $x \not\sqsubseteq \bigsqcup I$, contradiction. □

Definition 2.9 *A lattice is* **continuous** *if it is complete and, for all x,*

$$x = \bigsqcup \{a : a \ll x\}.$$

Thus a lattice is continuous if every element has enough way below it. Moreover, the following are continuous:

- every finite lattice;
- every complete linear ordering;
- every power set.

It follows that not every continuous lattice is distributive, and in particular *not every continuous lattice is a Heyting algebra*. On the other hand, lack of distributivity is the only obstruction, since *a distributive continuous lattice is a Heyting algebra* (as noticed after the proof of 2.14, a distributive continuous lattice has enough points; and as noticed after the proof of 1.12, a complete lattice with enough points is a Heyting algebra).

A spectrum of Heyting algebras

Having now introduced all notions of Heyting algebras we wanted to, it is time to pause for a moment to look at their mutual relationships. We first prove a series of results, showing inclusions.

Proposition 2.10 *Every finite Heyting algebra has enough strong co-points.*

Proof. We want to show that every non-zero element of a finite Heyting algebra is the l.u.b. of the strong co-points below it. In a finite lattice $\bigsqcup$ and $\sqcup$ coincide, and hence so do strong co-points and co-points; if in addition the lattice is distributive, then they both coincide with the $\sqcup$-irreducible elements. And in a finite lattice every non-zero element is obviously the l.u.b. of the $\sqcup$-irreducible elements below it. $\square$

Proposition 2.11 *Every complete Heyting algebra with enough strong co-points is algebraic.*

Proof. A complete Heyting algebra with enough strong co-points is algebraic because a strong co-point is compact: then if every element is the l.u.b. of the strong co-points below it, it is also the l.u.b. of the compact elements below it. $\square$

Proposition 2.12 *Every algebraic Heyting algebra is continuous.*

Proof. If a is a compact element such that $a \sqsubseteq x$ then $a \ll a \sqsubseteq x$, and hence $a \ll x$: then if every element is the l.u.b. of the compact elements below it (i.e. if the lattice is algebraic), it is also the l.u.b. of the elements way below it (i.e. the lattice is continuous). $\square$

Before proving the last result in the series of inclusion, namely that every continuous Heyting algebra has enough points, we prove the following weaker version.

Proposition 2.13 Papert [1959]) *Every complete Heyting algebra with enough strong co-points has enough points.*

Proof. Let $x \not\sqsubseteq y$. Since the strong co-points below x have l.u.b. x, there is a strong co-point a such that $a \sqsubseteq x$ and $a \not\sqsubseteq y$. We consider the set

$$F = \{z : a \sqsubseteq z\},$$

and notice the following properties:

- F is a filter. In particular, F is closed under upward and under $\sqcap$.
- F contains only elements above a. In particular $y \notin F$.
- No element of $\overline{F}$ is above x. This follows from the fact that F is upward closed.
- Every maximal element in $\overline{F}$ is $\sqcap$-irreducible, and hence a point. This follows from the fact that F is closed under $\sqcap$.
- If X is contained in $\overline{F}$, then $\bigsqcup X$ is also in $\overline{F}$. Otherwise $a \sqsubseteq \bigsqcup X$, and since a is a strong co-point there is $x \in X$ such that $a \sqsubseteq x$, i.e. $X \cap F \neq \emptyset$.

Let now C be any maximal chain in $\overline{F}$ containing y, which exists by Zorn's Lemma. Then $\bigsqcup C$ is also in $\overline{F}$. Since C is maximal, $\bigsqcup C$ is a maximal element of $\overline{F}$, and hence a point above y (by the choice of C), but not above in x (because it is in $\overline{F}$). $\square$

We now generalize the previous proof to continuous Heyting algebras.

Proposition 2.14 (Papert [1959]) *Every continuous Heyting algebra has enough points.*

Proof. We refer to the proof of 2.13: to be able to extend its last part, given $x \not\sqsubseteq y$ we need to find a filter F with the properties used there.

Since the elements way below x have l.u.b. x, there is $a \ll x$ such that $a \not\sqsubseteq y$. By density of $\ll$ (2.8), there is an infinite descending chain

$$a \ll \cdots \ll x_2 \ll x_1 \ll x.$$

We consider the set

$$F = \{z : (\exists n)(x_n \sqsubseteq z)\},$$

and notice the following properties:

- F is a filter, being a union of principal filters (generated by the x_n's). In particular, F is closed upward and under $\sqcap$.
- F contains only elements above a. In particular $y \notin F$.
- No element of $\overline{F}$ is above x. This follows from the fact that $x \in F$ (because $x_1 \ll x$, and hence $x_1 \sqsubseteq x$), since F is upward closed.
- Every maximal element in $\overline{F}$ is $\sqcap$-irreducible, and hence a point. This follows from the fact that F is closed under $\sqcap$.
- If I is an ideal contained in $\overline{F}$, then $\bigsqcup I$ is also in $\overline{F}$.[6] Otherwise $\bigsqcup I$ is in F, and there is some n such that $x_n \sqsubseteq \bigsqcup I$; since $x_{n+1} \ll x_n$, x_{n+1} must be below some element of I, which is impossible because I is contained in $\overline{F}$, and F is upward closed.

Let now C be any maximal chain in $\overline{F}$ containing y, which exists by Zorn's Lemma. The downward closure of C is an ideal I contained in $\overline{F}$, and thus $\bigsqcup I$ is also in $\overline{F}$. Since C is maximal, $\bigsqcup I$ is a maximal element of $\overline{F}$, and hence a point above y (by the choice of C), but not above in x (because it is in $\overline{F}$). $\square$

Notice that the proof shows in particular that *a distributive continuous lattice has enough points*, since no use of $\Rightarrow$ was made in it. On the other hand, distributivity (which was used when claiming that a $\sqcap$-irreducibile element is a point) is

[6] This property of F (of being inaccessible by l.u.b.'s of ideals in $\overline{F}$) is characteristic of open sets in the Scott topology.

essential, since the previous result implies, together with 1.12, that *every distributive continuous lattice is a Heyting algebra*, while not every continuous lattice is such (see the examples on p. 20).

We now turn to a series of result showing that the inclusions just proved are proper. We choose the counterexamples among the most natural mathematical structures and topologies, to show that the present spectrum of Heyting algebras allows to distinguish among them from a purely algebraic point of view.

Proposition 2.15 *The algebra $\mathcal{P}(\omega)$ of all sets of natural numbers is an infinite complete Heyting algebra with enough strong co-points.*

Proof. Every singleton is a strong co-point, and every set is the union of the singletons contained in it. □

Proposition 2.16 *The algebra of open sets of the Cantor space 2^ω is an algebraic Heyting algebra without (strong) co-points.*

Proof. Recall that the Cantor space is the set of all 0,1-valued functions on ω, with the topology generated by the sets $\{f : f \supseteq \sigma\}$ of functions having a common fixed initial segment σ.

The algebra of open sets of the Cantor space is algebraic because such sets are compact (by König's Lemma), and generate the topology by definition. And there is no co-point because each such set is the union of different open sets, since each function having initial segment σ must have as next value either 0 or 1:

$$\{f : f \supseteq \sigma\} = \{f : f \supseteq \sigma * 0\} \cup \{f : \supseteq \sigma * 1\}. \quad \Box$$

Proposition 2.17 *The algebra of open sets of the euclidean space $I\!R$ is a continuous Heyting algebra which is not algebraic.*

Proof. First of all, it is not algebraic because the only compact open set is $\emptyset$.

Secondly, we prove that if V is an open set and U is an open interval whose closure $cl(U)$ is contained in V, then $U \ll V$: continuity then follows from the fact that the open intervals generate the topology (more specifically, that every open set is a union of open intervals).

Suppose $V \subseteq \bigcup_{i \in I} A_i$, where $\mathcal{I} = \{A_i\}_{i \in I}$ is an ideal of open sets. Then

$$U \subseteq cl(U) \subseteq V \subseteq \bigcup_{i \in I} A_i.$$

Since $cl(U)$ is a closed interval, it is compact: then there are $i_1, \ldots, i_n$ such that

$$U \subseteq cl(U) \subseteq A_{i_1} \cup \cdots \cup A_{i_n}.$$

But $A_{i_1} \cup \cdots \cup A_{i_n} \in \mathcal{I}$ by closure under finite $\cup$, and then $U \in \mathcal{I}$ by downward closure of $\mathcal{I}$. □

Proposition 2.18 *The algebra of open sets of the Baire space ω^ω is a complete Heyting algebra with enough points which is not continuous.*

Proof. Recall that the Baire space is the set of all functions on ω, with the topology generated by the sets $\{f : f \supseteq \sigma\}$ of functions having a common fixed initial segment σ.

First of all, the algebra of open sets of the Baire space has enough points because so does every topology, by the proof of 1.12. Actually, in the present case one can show that, for any function f, the set $\overline{\{f\}}$ is a point: it is open because it is the union of the basic open sets defined by initial segments σ differing from f on at least one point; and it is a point because if $\overline{\{f\}} \supseteq A \cap B$ then at least one of A and B does not contain f, and it is thus contained in $\overline{\{f\}}$. Moreover, there are enough such points because if A and B are distinct open sets then there is a function f in one but not in the other, say $f \in A - B$: then the point $\overline{\{f\}}$ contains B but not A.

Secondly, to prove that such an algebra is not continuous it is enough to show that if A and B are open sets such that $A \ll B$, then $A = \emptyset$. Suppose $A \neq \emptyset$: it is enough to show that it is possible to decompose the whole space ω^ω into infinitely many disjoint open sets, each containing at least an element of A. If I is the ideal generated by such open sets, then $B \subseteq \bigcup I$: since $A \ll B$, A should be in I, and hence be contained in a finite union of such open sets, contradiction.

If $A \neq \emptyset$ then it contains a basic open set, defined by an initial segment σ. On the one hand, the complement of such open set is open (because it is the union of the basic open sets defined by initial segments incompatible with σ). On the other hand, such open set is the disjoint union of infinitely many basic open sets (the ones defined by one element extensions of σ). The needed decomposition of the whole space is easily obtained from these open sets. □

Proposition 2.19 *The algebra $\mathcal{P}^*(\omega)$ of all sets of natural numbers modulo finite sets is a complete Heyting algebra without points.*

Proof. There is no point because each equivalence class except the greatest one contains coinfinite sets, and if A is coinfinite then it is the intersection of two coinfinite sets A_1 and A_2 differing coinfinitely from A (e.g. $A_i = A \cup B_i$, where B_1 and B_2 are infinite disjoint sets such that $B_1 \cup B_2 = \overline{A}$). □

Locally quasi-compact topologies

We now conclude our treatment by characterizing the continuous Heyting algebras from a topological point of view.

By 2.3 the (dual) Stone topology of an arbitrary Heyting algebra is generated by compact open sets (since each $f(x)$ is compact). We now look at the dual Stone topology of a continuous Heyting algebra (as an algebra with enough points), and show that it is generated by the interiors of compact open sets, in the following sense.

Definition 2.20 *A topological space is called* **locally quasi-compact** *if, for any element x and any open set V such that $x \in V$, there are a compact set C_x and an open set O_x such that*

$$x \in O_x \subseteq C_x \subseteq V.$$

A topological space is called **locally compact** *if it is locally quasi-compact and T_2.*

A typical example of a locally (quasi-)compact, but not compact space is given by $I\!R$ with the usual topology.

Proposition 2.21 (Day and Kelly [1970]) *Every algebra of open sets of a locally quasi-compact topology is a continuous Heyting algebra.*

Proof. We generalize the fact, proved in 2.17, that the algebra of open sets of $I\!R$ is continuous.

Given an open set V, for any $x \in V$ we consider the compact set C_x and the open set O_x provided by the definition of local quasi-compactness. Since $V = \bigcup_{x \in V} O_x$, to prove that the algebra of open sets is continuous it is enough to show that $O_x \ll V$: then each open set is the l.u.b. of open sets way below it.

Suppose $V \subseteq \bigcup_{i \in I} A_i$, where $\mathcal{I} = \{A_i\}_{i \in I}$ is an ideal of open sets. Then

$$O_x \subseteq C_x \subseteq V \subseteq \bigcup_{i \in I} A_i.$$

By definition, from any family of open sets whose union covers a compact set, one can extract a finite subfamily with the same property. Then, by compactness of C_x,

$$O_x \subseteq C_x \subseteq A_{i_1} \cup \cdots \cup A_{i_n}$$

for some $i_1, \ldots, i_n$. But $A_{i_1} \cup \cdots \cup A_{i_n} \in \mathcal{I}$ by closure under finite $\cup$, and then $O_x \in \mathcal{I}$ by downward closure of $\mathcal{I}$. □

We turn now to the other half of the result.

Proposition 2.22 (Hofmann and Lawson [1978]) *A continuous Heyting algebra is isomorphic to the algebra of open sets of a locally quasi-compact topology.*

Proof. By 2.14 a continuous Heyting algebra has enough points, and by 1.12 it is isomorphic to the algebra of open sets of the form

$$p(x) = \{a : a \text{ point } \wedge \ a \not\supseteq x\}.$$

It is thus enough to show that such a topology is locally quasi-compact, i.e. that for any element y and any open set $p(x)$ such that $y \in p(x)$, there are a compact set C_y and an open set O_y such that

$$y \in O_y \subseteq C_y \subseteq p(x).$$

Since $y \in p(x)$, y is a point such that $x \not\subseteq y$. As in the proof of 2.14, there is $a \ll x$ such that $a \not\subseteq y$, and one can find a filter F with the following properties:

- F contains only elements $\sqsupseteq a$.
- No element of $\overline{F}$ is above x.
- If I is an ideal contained in $\overline{F}$, then $\bigsqcup I$ is also in $\overline{F}$.

Since y is a point and $y \not\sqsupseteq a$, $y \in p(a)$: but $p(a)$ is open, so we can let $O_y = p(a)$. Moreover, $p(a)$ is a set of points not above a, and hence in $\overline{F}$ and not above x: if we let C_y be the set of points in $\overline{F}$, we then automatically have

$$y \in O_y \subseteq C_y \subseteq p(x).$$

It only remains to show that C_y is compact, and we prove this by contrapositive.

First we notice that the downward closure of C_y coincides with $\overline{F}$. Indeed, if an element is below a point in $\overline{F}$ then it cannot be in F, because F is upward closed (being a filter), and hence its complement is downward closed. Conversely, any element of $\overline{F}$ is bounded by a point in $\overline{F}$, by the proof of 2.14.

Given now a subset B of A suppose that, for any finite subset $\{a_1, \ldots, a_n\}$ of B,

$$C_y \not\subseteq p(a_1) \cup \cdots \cup p(a_n).$$

Since p preserves $\sqcup$,

$$C_y \not\subseteq p(a_1 \sqcup \cdots \sqcup a_n).$$

This means that there is a point in $\overline{F}$ above $a_1 \sqcup \cdots \sqcup a_n$. Since $\overline{F}$ is the downward closure of C_y, this means that the ideal I generated by B is contained in $\overline{F}$: by the properties of F then $\bigsqcup I$ is in $\overline{F}$, and hence so is $\bigsqcup B$. Again because $\overline{F}$ is the downward closure of C_y, this means that $\bigsqcup B$ is bounded by an element in C_y, i.e. by a point in $\overline{F}$: then

$$C_y \not\subseteq p(\bigsqcup B).$$

Since p preserves $\bigsqcup$,

$$C_y \not\subseteq \bigcup_{a \in B} p(a). \quad \square$$

We can now put the two halves together.

Theorem 2.23 Topological Characterization of Continuous Heyting Algebras (Day and Kelly [1970], Hofmann and Lawson [1978]) *The continuous Heyting algebras are, up to isomorphism, exactly the algebras of open sets of locally quasi-compact topologies.*

Proof. By 2.21 and 2.22. $\square$

By describing topologies with reference only to their open sets, and not to the points of their underlying spaces, 1.12 allows one to see the theory of Heyting algebras as a kind of *pointless topology*, in which topological properties can be described in a purely algebraic way. 2.23 provides a paradigm in this direction, capturing the topological property of local quasi-compactness in terms of the algebraic property of continuity.

Conclusion

We summarize the series of results proved in the two lectures in the following table.

Heyting algebras	**topologies**	**typical examples**
finite	finite	
enough strong co-points	closed under intersection	$\mathcal{P}(\omega)$
algebraic	generated by compact opens	2^ω (Cantor space)
continuous	locally quasi-compact	$\mathbb{R}$ (Euclidean space)
enough points	arbitrary	ω^ω (Baire space)
arbitrary		$\mathcal{P}^*(\omega)$

Bibliography

Birkhoff, G., and Frink, O.

[1948] Representation of lattices by sets, *Trans. Am. Math. Soc.* 64 (1948) 299–316.

Büchi, J.R.

[1952] Representation of complete lattices by sets, *Portug. Math.* 11 (1952) 151–167.

Day, B.J., and Kelly, G.M.

[1970] On topological quotients maps preserved by pullbacks or products, *Proc. Cambr. Phil. Soc.* 67 (1970) 553–558.

Gentzen, G.

[1935] Untersuchungen über das logische Schliessen, *Math. Zeit.* 39 (1935) 176–210, 405–431, transl. in [1969], pp. 68–131.

[1969] *Collected papers*, North Holland, 1969.

Hofmann, K.H., and Keimel, K.

[1972] *A general character theory for partial ordered sets and lattices*, Memoirs of the American Mathematical Society n. 122, 1972.

Hofmann, K.H., and Lawson, J.D.

[1978] The spectral theory of distributive continuous lattices, *Trans. Amer. Math. Soc.* 246 (1978) 285–310.

McKinsey, J.C.C., and Tarski, A.

[1946] On closed elements in closure algebras, *Ann. Math.* 47 (1946) 122–162.

Nachbin, L.

[1949] On a characterization of the lattice of all ideals of a Boolean ring, *Fund. Math.* 36 (1949) 137–142.

Papert, S.

[1959] Which distributive lattices are lattices of closed sets?, *Proc. Camb. Phil. Soc.* 55 (1959) 172–176.

Raney, G.N.

[1952] Completely distributive complete lattices, *Proc. Amer. Math. Soc.* 3 (1952) 677–680.

Scott, D.S.

[1972] Continuous lattices, *Springer Lect. Not. Math.* 274 (1972) 97–136.

Stone, M.H.

[1937] Topological representation of distributive lattices and Brouwerian logic, *Čas. Mat. Fys.* 67 (1937) 1–25.

[1937a] Applications of the theory of Boolean rings to general topology, *Trans. Am. Math. Soc.* 41 (1937) 321–364.