INDUCTIVE INFERENCE OF TOTAL FUNCTIONS

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In this paper we give an overview of an area of applied Recursion Theory that has attracted interest in recent years. The idea is to use recursion-theoretic notions to formalize the venerable problem of inductive inference, in the following way.

By taking time to be discrete and with a starting point, and events to be discrete and codifiable by natural numbers, a phenomenon to be inferred may be thought of as a function f on the natural numbers, given by the sequence of values

$$f(0), \ldots, f(n), \ldots$$

The function can be inferred if this is not just a sequence of accidents, but rather it has an intrinsic necessity. We can specify this internal structure of the sequence of values in at least two ways:

- If one is interested in tecnology, i.e. in the ability of reproducing effects, then one can require a method predicting the next value $f(n+1)$, once the values $f(0), \ldots, f(n)$ have been exhibited, for an arbitrary n.
- If one is interested in science, i.e. in the ability of understanding causes, then one can require a finite description compressing the infinite amount of information contained in the sequence of values.

Even in this vague formulation, one can identify the functions which are (in principle) inferrable w.r.t. any of the two methods with the recursive functions.

This completely disposes of the problem of which functions on natural numbers are *individually* inferrable, and one can thus turn the attention to *classes* of functions. The problem here takes the following form: for each class $\mathcal{C}$ of recursive functions, find a uniform method of inferring all members of $\mathcal{C}$.

Many possible formalizations of notions of inference for classes of total recursive functions have been considered in the literature. Here we will confine ourselves to a few of them, and refer to the forthcoming volume II of our book *Classical Recursion Theory* (Odifreddi [1997]) for a comprehensive treatment. For background and notations we refer instead to volume I of the same book (Odifreddi [1989]).

1 Identification by next value

Our first notion formalizes the idea of uniform method of prediction or extrapolation.

Definition 1.1 (Barzdin [1972], Blum and Blum [1975]) *A class $\mathcal{C}$ of total recursive functions is* **identifiable by next value** *($\mathcal{C} \in \boldsymbol{NV}$) if there is a total recursive function g (called a next-value function for $\mathcal{C}$) such that, for every $f \in \mathcal{C}$ and almost every n,*

$$f(n+1) = g(\langle f(0), \ldots, f(n) \rangle).$$

Notice that we allow a finite number of wrong predictions for each element of the class, i.e. g can take guesses and learn from its mistakes.

Theorem 1.2 Number-Theoretic Characterization of $\boldsymbol{NV}$ (Barzdin and Freivalds [1972]) *A class of total recursive function is in NV if and only if it is a subclass of an r.e. class of total recursive functions.*

Proof. A next-value function g allows the computation of a recursive function f, past the finitely many exceptions. Thus any function f for which g is a next-value function is of the following form, for some sequence number a (coding a list $\langle a_0, \ldots, a_n \rangle$ for some n):

$$f_a(x) = \begin{cases} a_x & \text{if } x \leq n \\ g(\langle f_a(0), \ldots, f_a(x-1) \rangle) & \text{otherwise.} \end{cases}$$

Any such f_a is recursive (by Course-of-Value Recursion) uniformly in a, and by the S^m_n-Theorem there is then a recursive function h such that $\varphi_{h(a)} = f_a$. Thus the class $\{f_a\}_{a \in \omega}$ is an r.e. class of total recursive functions. This shows that any class of recursive functions identifiable by next value is a subclass of an r.e. class of total recursive functions.

Conversely, for an r.e. class $\{\varphi_{h(e)}\}_{e \in \omega}$ of total recursive functions, we may suppose it closed under finite variants (since the closure of an r.e. class under finite variants is still r.e.). Let g be the recursive function defined as follows:

- on the empty list, g takes the value 0;
- on the list $\langle a_0, \ldots, a_n \rangle$, g takes the value $\varphi_{h(e)}(n+1)$, for the first e such that $\varphi_{h(e)}(x) = a_x$ for all $x \leq n$ (i.e. g takes the next value of the first function in the class that agrees with all values coded by the given list).

Since the class $\{\varphi_{h(e)}\}_{e \in \omega}$ is closed under finite variants, g is total. Since the class is r.e. (i.e. the functions $\varphi_{h(e)}$ are uniformly recursive), g is recursive. And g is a next-value function for every function in the given class (and hence in any subclass of it), by definition. $\Box$

Theorem 1.3 Complexity-Theoretic Characterization of NV (Adleman) *A class of total recursive functions is in NV if and only if it is a subclass of a complexity class (w.r.t. some complexity measure).*

Proof. By 1.2, any class in NV is contained in an r.e. class of total recursive functions $\mathcal{C} = \{\varphi_{h(e)}\}_{e\in\omega}$. Consider the associated set $\{\Phi_{h(e)}\}_{e\in\omega}$ of step-counting functions w.r.t. any measure, and notice that if

$$t(x) = \max_{e\leq x} \Phi_{h(e)}(x)$$

then t is recursive because h is, and $\mathcal{C}$ is contained in the complexity class $\mathcal{C}_t$ named by t.

Conversely, given a recursive function t, notice that if φ_e is total and

$$(\forall_\infty x)[\Phi_e(x) \leq t(x)]$$

then there is a constant k such that

$$(\forall x)[\Phi_e(x) \leq t(x) + k].$$

Let g be the recursive function defined as follows:

- on the empty list, g takes the value 0
- on the list $\langle a_0, \ldots, a_n\rangle$, g takes the value $\varphi_e(n+1)$ for the smallest pair $\langle e, k\rangle$ defined as follows, if there is one, and 0 otherwise:
  - $\langle e, k\rangle \leq n$
  - for any $x \leq n+1$, $\Phi_e(x) \leq t(x) + k$
  - for any $x \leq n$, $\varphi_e(x) \simeq a_x$.

g is total recursive because all checks are recursive, and the second condition on $\langle e, k\rangle$ ensures that $\varphi_e(n+1)$ is defined.

It is easy to show that g identifies by next value the complexity class $\mathcal{C}_t$. □

2 Identification by consistent explanation

We now turn to notions that formalize the idea of uniform method of explanation (via indices, that code descriptions of recursive functions). In a first attempt we require that the explanations agree with the available data.

Definition 2.1 (Gold [1967]) *A class $\mathcal{C}$ of total recursive functions is* **identifiable by consistent explanation** *($\mathcal{C} \in \boldsymbol{EX_{cons}}$) if there is a total recursive function g (called a guessing function for $\mathcal{C}$) such that, for every sequence number $\langle a_0, \ldots, a_n\rangle$:*

- $\varphi_{g(\langle a_0,\ldots,a_n\rangle)}(x) \simeq a_x$ *for all* $x \leq n$,

and for every $f \in \mathcal{C}$:

- $\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)$ *exists, i.e.*

$$(\exists n_0)(\forall n \geq n_0)[g(\langle f(0),\ldots,f(n)\rangle) = g(\langle f(0),\ldots,f(n_0)\rangle)]$$

- $\varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)} = f$.

In other words, $g(\langle f(0),\ldots,f(n)\rangle)$ *provides a guess to an index of* f *consistent with the available information, and the guess stabilizes (from a certain point on) on an index of* f.

The next result connects the two notions of identification introduced so far, and its two proofs suggest the two characterizations of EX_{cons} given in 2.5 and 2.7.

Proposition 2.2 (Gold [1967]) $NV \subseteq EX_{cons}$.

Proof. We can use the characterization of NV given in 1.2, and repeat almost verbatim the second half of its proof.

Alternatively, we can use the characterization of NV given in 1.3, and again repeat almost verbatim the second half of its proof. □

The difficulty in proving the opposite inclusion is the following: given a guessing function g for a class of functions $\mathcal{C}$, one might think of producing as a next-value function for $\mathcal{C}$ the one defined by

$$g_1(\langle a_0,\ldots,a_n\rangle) = \varphi_{g(\langle a_0,\ldots,a_n\rangle)}(n+1),$$

i.e. to let the guessed program guess the next value. The problem with this is that $g(\langle a_0,\ldots,a_n\rangle)$ might be a program for a partial function (since we only know that the *final* guesses on initial segments of functions in $\mathcal{C}$ will be programs for total functions), and thus g_1 might not be total.

If one restricts the guesses to programs for total functions, then the problem disappears. One can argue that such a restriction is implicit in Popper Refutability Principle (Popper [1934]), according to which incorrect scientific explanations should be refutable: the unsolvability of the Halting Problem makes it in general impossible to decide whether a partial function is undefined at a given argument, and thus to refute an exlanation which is incorrect on such a basis. For more on this, see the discussion following 5.2.

We now show that if g is allowed to output indices for partial functions, then one is able to identify by consistent explanation more classes of functions. The result may be taken to show that technique has stronger requirements than science, and that being able to eventually explain a class of phenomena is not enough to be able to eventually predict them.

Proposition 2.3 (Blum and Blum [1975]) $EX_{cons} - NV \neq \emptyset$.

Proof. We want to find a class $\mathcal{C}$ of total recursive functions which is in EX_{cons} but not in NV. Let

$$\mathcal{C} = \{\Phi_e : \Phi_e \text{ total}\},$$

i.e. the class of all total step-counting functions w.r.t. any measure.

To show $\mathcal{C} \in EX_{cons}$ we can note that $\mathcal{C}$ is contained in the class $\{\Phi_e\}_{e \in \omega}$, which is a measured set of partial recursive functions, and prove the following result, of independent interest: *any class of total recursive functions contained in a measured set of partial recursive functions is in* EX_{cons}. The proof is a trivial extension of the first proof of 2.2.

Alternatively, to show $\mathcal{C} \in EX_{cons}$ we can note that $\mathcal{C}$ is contained in the class $\{\Phi_e\}_{e \in \omega}$, which is a honest set of partial recursive functions, and prove the following result, of independent interest: *any class of total recursive functions contained in a honest set of partial recursive functions is in* EX_{cons}. The proof is a trivial extension of the second proof of 2.2. $\square$

We turn now to a characterization of EX_{cons}. The characterization of NV given in 1.2 already used up all r.e. classes of total recursive functions, and thus we will look at r.e. classes of *partial* recursive functions.

We consider a notion that isolates just what is needed to make the proofs of 2.2 and 2.3 work.

Definition 2.4 *An r.e. class* $\{\varphi_{h(e)}\}_{e \in \omega}$ *is called* **quasi-measured** *if there is a uniform recursive procedure to decide, given any* e *and any finite initial segment* σ*, whether* $\varphi_{h(e)}$ *extends* σ*.*

Obviously, a measured set is quasi-measured: to check whether $\varphi_{h(e)}$ extends σ, one simply checks whether $\varphi_{h(e)}(x) \simeq \sigma(x)$ for every x in the domain of σ.

But the converse does not hold: quasi-measuredness allows us to check whether $\varphi_{h(e)}$ extends or not a given finite initial segment, but not whether it has a certain value on an isolated argument (unless we already know the values on all previous arguments).

Theorem 2.5 Number-Theoretic Characterization of $\boldsymbol{EX_{cons}}$ (Viviani) *A class of total recursive functions is in* EX_{cons} *if and only if it is a subclass of a quasi-measured set of partial recursive functions.*

Proof. The first proof of 2.3, showing that any class of total recursive functions contained in a measured set of partial recursive functions is in EX_{cons}, can easily be adapted to quasi-measured sets.

Conversely, let $\mathcal{C}$ be identifiable by consistent explanation via g. Then every function f in $\mathcal{C}$ is of the following form, for some sequence number a (coding a list $\langle a_0, \ldots, a_n \rangle$ for some n):

$$f_a(x) \simeq \begin{cases} a_x & \text{if } x \leq n \\ \varphi_{g(\langle a_0,\ldots,a_n\rangle)}(x) & \text{if } x > n \text{ and } g \text{ is not forced to change} \\ \text{undefined} & \text{otherwise} \end{cases}$$

(more precisely, $f \in \mathcal{C}$ is equal to f_a for any sequence number a coding the first n values of f, for any n such that $g(\langle f(0), \ldots, f(n) \rangle)$ has reached its limit). Any such f_a is partial recursive uniformly in a, and by the S^m_n-Theorem there is then a recursive function h such that $\varphi_{h(a)} = f_a$. Thus the class $\{f_a\}_{a\in\omega}$ is an r.e. class of partial recursive functions containing $\mathcal{C}$.

It remains to show that $\{f_a\}_{a\in\omega}$ is quasi-measured. Given a sequence number $a = \langle a_0, \ldots, a_n \rangle$ and an initial segment σ of length $m+1$, f_a extends σ if and only if:

1. either σ is contained in a (as a partial function), i.e. $\sigma(x) \simeq a_x$ for all $x \leq m$;

2. or σ extends a and $g(\langle a_0, \ldots, a_n \rangle)$ does not change on σ, i.e.

$$\begin{aligned} g(\langle a_0, \ldots, a_n \rangle) &= g(\langle a_0, \ldots, a_n, \sigma(n+1) \rangle) \\ &\cdots \\ &= g(\langle a_0, \ldots, a_n, \sigma(n+1), \ldots, \sigma(m) \rangle). \end{aligned}$$

The condition is obviously necessary, by definition of f_a, and we now show that it is also sufficient. In case 1, f_a agrees with a up to n, and hence with σ up to m. In case 2, f_a agrees with a, and hence with σ, up to n. For $n+1$, notice that

$$\varphi_{g(\langle a_0,\ldots,a_n\rangle)}(n+1) = \varphi_{g(\langle a_0,\ldots,a_n,\sigma(n+1)\rangle)}(n+1) = \sigma(n+1),$$

where the first equality holds because $g(\langle a_0, \ldots, a_n \rangle)$ does not change on σ, and the second does by consistency of g; this shows in particular that $f_a(n+1)$ is defined, because g is not forced to change, and that

$$f_a(n+1) = \sigma(n+1).$$

The proof for $n+2$ is similar, using the fact just proved that $f_a(n+1) = \sigma(n+1)$, and that g does not change on σ by hypothesis. In the same way one can proceed all the way to m. □

As we have already done for the notion of measuredness in 2.4, we now consider a weakening of the notion of honesty. Recall that f is h-honest if

$$(\exists e)[f \simeq \varphi_e \ \wedge \ (\forall_\infty x)(\Phi_e(x) \leq h(x, \varphi_e(x))].$$

Definition 2.6 *If f and h are recursive functions, then f is* **quasi-h-honest** *if*

$$(\exists e)[f \simeq \varphi_e \ \wedge \ (\forall_\infty x)(\Phi_e(x) \leq h(x, \max_{y\leq x} \varphi_e(y)))].$$

Obviously, if f is h-honest and h is monotone then f is quasi-h-honest:

$$\Phi_e(x) \leq h(x, \varphi_e(x)) \leq h(x, \max_{y \leq x} \varphi_e(y)).$$

But the converse does not hold: quasi-h-honesty provides a bound to $\Phi_e(x)$ in terms only of $\max_{y \leq x} \varphi_e(y)$, not just of $\varphi_e(x)$.

Theorem 2.7 Complexity-Theoretic Characterization of $\boldsymbol{EX_{cons}}$. *A class of total recursive functions is in EX_{cons} if and only if it is a class of quasi-h-honest functions, for some recursive function h.*

Proof. The second proof of 2.3, showing that any class of total recursive functions contained in a set of honest functions is in EX_{cons}, can easily be adapted to quasi-honest sets, by substituting $h(x, \max_{y \leq x} a_y)$ for $h(x, a_x)$.

Conversely, let $\mathcal{C}$ be identifiable by consistent explanation via g. Then one can define the following function:

$$h(x, z) = \max\{\Phi_{g(\langle a_0, \ldots, a_x \rangle)}(x) : z = \max_{y \leq x} a_y\}.$$

Since g is consistent, $\varphi_{g(\langle a_0, \ldots, a_x \rangle)}(x) \simeq a_x$, and so $\Phi_{g(\langle a_0, \ldots, a_x \rangle)}(x)$ is defined; moreover, there are only finitely many sequence numbers $\langle a_0, \ldots, a_x \rangle$ such that $z = \max_{y \leq x} a_y$, and hence h is total recursive.

If $f \in \mathcal{C}$ then g has a limit e on f, and $\varphi_e \simeq f$, because g identifies $\mathcal{C}$. If x_0 is a point after which g does not change anymore on f, then

$$\Phi_e(x) \leq h(x, \max_{y \leq x} f(y))$$

for all $x \geq x_0$. Thus e is a witness of the fact that f is quasi-h-honest. □

3 Identification by reliable explanation

The next notion formalizes the idea of a method of explanation that never permanently settles on a false hypothesis, and thus gives indirect information about its mistakes.

Definition 3.1 (Blum and Blum [1975]) *A class $\mathcal{C}$ of total recursive functions is* **identifiable by reliable explanation** *($\mathcal{C} \in \boldsymbol{EX_{rel}}$) if there is a total recursive function g such that, for every $f \in \mathcal{C}$:*

- $\lim_{n \to \infty} g(\langle f(0), \ldots, f(n) \rangle)$ *exists, i.e.*

$$(\exists n_0)(\forall n \geq n_0)[g(\langle f(0), \ldots, f(n) \rangle) = g(\langle f(0), \ldots, f(n_0) \rangle)]$$

and for every total recursive f:[1]

[1] This is the most we can ask in our setting, since $g(\langle f(0), \ldots, f(n) \rangle)$ can be defined for all n only if f is total, and its limit can be an index of f only if f is recursive.

- $\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)$ *exists* $\Rightarrow$ $\varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)} = f$.

In other words, $g(\langle f(0),\ldots,f(n)\rangle)$ provides a guess to an index of f, which stabilizes (from a certain point on) on an index of f whenever it stabilizes, and it does stabilize if $f \in \mathcal{C}$.

Notice that, despite the fact that we do not require from our guesses that they be consistent with the available information, such information cannot be disregarded, lest we proceed independently of f and never be able to stabilize our guess.

Proposition 3.2 *$EX_{cons} \subseteq EX_{rel}$.*

Proof. Let g identify $\mathcal{C}$ by consistent explanation. It is enough to show that

$$\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle) \text{ exists } \Rightarrow \varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)} = f.$$

If $\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)$ exists, let n_0 be such that

$$(\forall n \geq n_0)[g(\langle f(0),\ldots,f(n)\rangle) = g(\langle f(0),\ldots,f(n_0)\rangle)].$$

For every x, if n is greater than both n_0 and x then

$$\varphi_{g(\langle f(0),\ldots,f(n_0)\rangle)}(x) \simeq \varphi_{g(\langle f(0),\ldots,f(n)\rangle)}(x) \simeq f(x),$$

where the first equality holds because $n \geq n_0$, and the second one because $n \geq x$ and g is consistent. $\Box$

Having showed that reliable identification is at least as powerful as consistent identification, we now show that it is strictly more powerful.

Proposition 3.3 (Blum and Blum [1975], Fulk [1988]) *$EX_{rel} - EX_{cons} \neq \emptyset$.*

Proof. We want to find a class $\mathcal{C}$ of total recursive functions which is in EX_{rel} but not in EX_{cons}. Let

$$\mathcal{C} = \{f : (\exists e)[f \simeq \varphi_e \ \wedge \ (\forall x)(\Phi_e(x) \leq f(x+1))]\}.$$

To show that $\mathcal{C} \in EX_{rel}$, define g as follows: $g(\langle\ \rangle) = 0$; given $\langle a_0,\ldots,a_n\rangle$, look for the smallest $e \leq n$ such that $\Phi_e(x) \leq a_{x+1}$ and $\varphi_e(x) \simeq a_x$ for all $x < n$, and let $g(\langle a_0,\ldots,a_n\rangle)$ be such an e if one exists, and n otherwise.[2]

[2] Notice that this does not even imply that $\varphi_e(n)$ is defined, let alone that it is equal to a_n; thus the proposed identification procedure is not necessarily consistent (by the second part of the proof, it is provably not consistent).

To show that $\mathcal{C} \notin EX_{cons}$, suppose g is a consistent recursive guessing function such that

$$f \in \mathcal{C} \;\Rightarrow\; f = \varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)}.$$

The idea is to construct $f \in \mathcal{C}$ such that either $\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)$ does not exist (because g changes its guess infinitely often), or it exists but it is not an index of f (because f differs from it on some argument).

By the S^m_n-Theorem, there is a recursive function t such that $\varphi_{t(e)}$ is defined as follows. If $x = 0$, then $\varphi_{t(e)}(0) \simeq 0$. If $x > 0$, wait until all of $\varphi_e(0),\ldots,\varphi_e(x-1)$, and hence also $\Phi_e(x-1)$, are defined. There are two possible cases:

1. If $g(\langle \varphi_e(0),\ldots,\varphi_e(x-1)\rangle) \neq g(\langle \varphi_e(0),\ldots,\varphi_e(x-1), \Phi_e(x-1)+1\rangle)$ then we define $\varphi_{t(e)}(x) \simeq \Phi_e(x-1)+1$, thus ensuring that g has changed its value once.

2. Otherwise, we define $\varphi_{t(e)}(x) \simeq \Phi_e(x-1)$, thus ensuring that $\varphi_{t(e)}$ is different from the function guessed by g on the initial segment of $\varphi_{t(e)}$ of length x.

By the Fixed-Point Theorem, there is e such that $\varphi_e \simeq \varphi_{t(e)}$. Let $f \simeq \varphi_e$: then f is total by induction, and in $\mathcal{C}$ because $\Phi_e(x) \leq f(x+1)$ for every x. There are now two possible cases:

- $\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)$ *does not exist*
  Then g does not identify f by consistent explanation.

- $\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)$ *exists*
  If $g(\langle f(0),\ldots,f(x)\rangle)$ is this limit, then case 1 cannot take place at x, otherwise g would have changed its value at least once more, and so

  $$\varphi_{g(\langle f(0),\ldots,f(x-1)\rangle)}(x) \simeq \varphi_{g(\langle f(0),\ldots,f(x-1),\Phi_e(x-1)+1\rangle)}(x) \simeq \Phi_e(x-1)+1,$$

  where the second equality holds by consistency of g. But since case 1 does not take place, $f(x) = \Phi_e(x-1)$; thus f differs from $\varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)}$ at x. $\Box$

We do not know of any number-theoretic characterization of EX_{rel}. For a complexity-theoretic characterization, a hint comes from the notion of quasi-honesty used in 2.7, and the observation that the expression

$$h(x, \max_{y\leq x} \alpha(y)),$$

used in it defines a partial recursive functional $H(\alpha, x)$.

Definition 3.4 *If f is a recursive function and H is a partial recursive functional, then f is **H-honest** if*

$$(\exists e)[f \simeq \varphi_e \;\wedge\; (\forall_\infty x)(H(\varphi_e, x)\downarrow \wedge\, \Phi_e(x) \leq H(\varphi_e, x))].$$

Theorem 3.5 Complexity-Theoretic Characterization of $\boldsymbol{EX_{rel}}$ (Blum and Blum [1975]) *A class $\mathcal{C}$ of total recursive functions is in EX_{rel} if and only if it is a class of H-honest functions, for some partial recursive functional H which is total on all total recursive functions.*

Proof. Given a partial recursive functional H total on all total recursive functions, notice that if φ_e is total and H-honest then there is a constant k such that

$$(\forall x)[\Phi_e(x) \leq H(\varphi_e, x) + k].$$

One can define a recursive function g that reliably identifies the class of all total H-honest functions, as follows:

- on the empty list, g takes the value 0
- on the list $\langle a_0, \ldots, a_n \rangle$, g takes the value e for the smallest pair $\langle e, k \rangle$ defined as follows, if there is one, and n otherwise:
  - $\langle e, k \rangle \leq n$
  - for all $x \leq n$, if $H(\langle a_0, \ldots, a_n \rangle, x)$ converges in at most n steps (where $\langle a_0, \ldots, a_n \rangle$ is considered as the partial function giving value a_x to $x \leq n$, and undefined otherwise) then

$$\Phi_e(x) \leq H(\langle a_0, \ldots, a_n \rangle, x) + k \qquad \text{and} \qquad \varphi_e(x) \simeq a_x.$$

It is easy to show that g identifies by reliable explanation the class of all total H-honest functions.

In the opposite direction, let $\mathcal{C}$ be identifiable by reliable explanation via g, and define

$$H(\alpha, x) \simeq \begin{cases} \Phi_{g(\langle \alpha(0), \ldots, \alpha(x) \rangle)}(x) & \text{if } \varphi_{g(\langle \alpha(0), \ldots, \alpha(x) \rangle)}(x) \text{ converges first} \\ 0 & \text{if } g \text{ changes its mind on } \alpha \text{ after } x \text{ first} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

H is a partial recursive functional by definition.

If f is a total recursive function, we show that $H(f, x)$ is defined for any x, so that H is total on the total recursive functions. Either $g(\langle f(0), \ldots, f(x) \rangle)$ changes after x, and then $H(f, x)$ is defined by the second clause if not otherwise; or $g(\langle f(0), \ldots, f(x) \rangle)$ is the limit of g on f and hence, by reliability, an index of the total function f, so that $H(f, x)$ is defined by the first clause.

Finally, if $f \in \mathcal{C}$ then g has a limit e on f, and $\varphi_e \simeq f$, because g identifies $\mathcal{C}$. If x_0 is a point after which g does not change anymore on f, then $H(f, x) \simeq \Phi_e(x)$ for all $x \geq x_0$, because $\varphi_e(x)$ converges (since f is total). Thus e is a witness of the fact that f is H-honest. $\square$

4 Identification by explanation

There is an obvious way of relaxing the definitions of EX_{cons} or EX_{rel}: just drop any consistency or reliability requirement.

Definition 4.1 (Gold [1967]) *A class $\mathcal{C}$ of total recursive functions is* **identifiable by explanation** *($\mathcal{C} \in \boldsymbol{EX}$) if there is a total recursive function g such that, for every $f \in \mathcal{C}$:*

- $\lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle)$ *exists, i.e.*

$$(\exists n_0)(\forall n \geq n_0)[g(\langle f(0),\dots,f(n)\rangle) = g(\langle f(0),\dots,f(n_0)\rangle)]$$

- $\varphi_{\lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle)} = f$.

In other words, $g(\langle f(0),\dots,f(n)\rangle)$ provides a guess to an index of f, and the guess stabilizes (from a certain point on) on an index of f.

Proposition 4.2 (Barzdin [1974], Blum and Blum [1975]) $EX_{rel} \subset EX$.

Proof. The inclusion is obvious by definition. To show that it is proper, we want to find a class $\mathcal{C}$ of total recursive functions which is in EX but not in EX_{rel}. Let

$$\mathcal{C} = \{f : f = \varphi_{f(0)}\},$$

i.e. the class of total recursive functions such that $f(0)$ is an index of f.

$\mathcal{C} \in EX$, by letting $g(\langle\ \rangle) = 0$ and $g(\langle a_0,\dots,a_n\rangle) = a_0$. In other words, every function in $\mathcal{C}$ gives away a program for itself as its first value, thus making identification by explanation trivial.

To show that $\mathcal{C} \notin EX_{rel}$, suppose g is a reliable recursive guessing function such that

$$f = \varphi_{f(0)} \Rightarrow f = \varphi_{\lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle)}.$$

By reliability, for any f

$$\lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle) \text{ exists} \Rightarrow f = \varphi_{\lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle)}.$$

Thus we cannot construct an $f \in \mathcal{C}$ such that if $\lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle)$ exists then it is not an index of f (as was one option in the proof of 3.3), and we are forced to find a function f such that

$$f = \varphi_{f(0)} \quad \text{and} \quad \lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle) \text{ does not exist.}$$

By the S^m_n-Theorem, there is a recursive function t such that $\varphi_{t(e)}$ is defined as follows. Start with $\varphi_{t(e)}(0) \simeq e$. Consider the functions f_i $(i = 0, 1)$ equal to e for $x = 0$, and identically equal to i for $x > 0$: of the two limits

$$\lim_{n\to\infty} g(\langle f_1(0),\dots,f_1(n)\rangle) \quad \text{and} \quad \lim_{n\to\infty} g(\langle f_2(0),\dots,f_2(n)\rangle)$$

either at least one does not exist, or they both exist and are different (by reliability of g, since the two functions f_0 and f_1 are different). Thus there must exist n and i ($i = 0$ or $i = 1$) such that

$$g(\langle \varphi_{t(e)}(0) \rangle) \neq g(\langle f_i(0), \ldots, f_i(n) \rangle),$$

and we can effectively find them. Let $\varphi_{t(e)}(x)$ be equal to $f_i(x)$ for every $x \leq n$, and iterate the procedure (by extending, at each step, the values already obtained by a sequence of either 0's or 1's).

By the Fixed-Point Theorem, there is e such that $\varphi_e \simeq \varphi_{t(e)}$. Let $f \simeq \varphi_e$: then f is in $\mathcal{C}$ because its value on 0 is an index for it, and g does not identify f because, by construction, $\lim_{n\to\infty} g(\langle f(0), \ldots, f(n) \rangle)$ does not exist. □

Notice that the class $\{f : f = \varphi_{f(0)}\}$ used in the previous proof is a subclass of the r.e. class of partial recursive functions $\{\varphi_{h(e)}\}_{e\in\omega}$ defined as follows:

$$\varphi_{h(e)}(x) \simeq \begin{cases} e & \text{if } x = 0 \\ \varphi_e(x) & \text{otherwise.} \end{cases}$$

We can effectively tell apart all pairs of members of such a class, since they all differ on their first arguments.

We turn now to a characterization of EX, in terms of the following generalization of the property just noticed.

Definition 4.3 *An r.e. class $\{\varphi_{h(e)}\}_{e\in\omega}$ is called* **effectively separable** *if there is a uniform recursive procedure to determine, for every different e and i, an upper bound to an argument on which $\varphi_{h(e)}$ and $\varphi_{h(i)}$ disagree.*

Theorem 4.4 First Number-Theoretic Characterization of *EX* (Wiehagen and Jung [1977], Wiehagen [1978]) *A class $\mathcal{C}$ of total recursive functions is in EX if and only if it is a subclass of an effectively separable r.e. class of partial recursive functions.*

Proof. Given an effectively separable r.e. class $\{\varphi_{h(e)}\}_{e\in\omega}$, there is a recursive function d (for 'disagreement') such that, for every different e and i, there is $x \leq d(e,i)$ such that $\varphi_{h(e)}(x) \not\simeq \varphi_{h(i)}(x)$. One can define a recursive function g that identifies by explanation any subclass $\mathcal{C}$ of the given class, as follows:

- on the empty list, g takes the value $h(0)$
- suppose the value of g on the list $\langle a_0, \ldots, a_{n-1} \rangle$ was $h(e)$; then on the list $\langle a_0, \ldots, a_n \rangle$ the value of g is still $h(e)$, unless there is an i as follows, in which case g takes the value $h(e+1)$:
  - i is different from e
  - $i \leq n$

- $d(e, i) \leq n$
- for all $x \leq d(e, i)$, $\varphi_{h(i)}(x) \simeq a_x$, with all computations convergent in at most n steps.

It is easy to show that $\mathcal{C}$ is identified by explanation via g.

In the opposite direction, let $\mathcal{C}$ be identifiable by explanation via g. Then every function in $\mathcal{C}$ is of the following form, for some sequence number a (coding a list $\langle a_0, \ldots, a_n \rangle$ for some n):

$$f_a(x) \simeq \begin{cases} a_x & \text{if } x \leq n \\ \varphi_{g(\langle a_0, \ldots, a_n \rangle)}(x) & \text{otherwise} \end{cases}$$

(more precisely, $f \in \mathcal{C}$ is equal to f_a for any sequence number a coding the first n values of f, for any n such that $g(\langle f(0), \ldots, f(n) \rangle)$ has reached its limit). Any such f_a is partial recursive uniformly in a, and by the S_n^m-Theorem there is then a recursive function h such that $\varphi_{h(a)} = f_a$. Thus the class $\{f_a\}_{a \in \omega}$ is an r.e. class of partial recursive functions containing $\mathcal{C}$. But not all members of it are different (since, by the parenthetical remark above, each function in $\mathcal{C}$ appears infinitely often in the class).

Incompatible sequence numbers are not problematic, since the associated functions agree with them, and hence they differ among each other. Among the compatible sequence numbers, we can certainly cut down a number of repetitions, by considering only those for which g provides new guesses: they form an r.e. class, and hence so does the set of associated functions. But we can still have two compatible sequence numbers a and b such that $f_a = f_b$: for example, both $g(a)$ and $g(b)$ might be the final guess of an index of a function in $\mathcal{C}$, but in between them g could have changed its mind.

We thus take two complementary actions: on the one hand, we start defining a new function only when we hit a sequence number that provides a new guess of g, and on the other hand we stop defining the new function as soon as we discover that the guess of g changes. In other words, for any sequence number $a = \langle a_0, \ldots, a_n \rangle$ such that

$$g(\langle a_0, \ldots, a_{n-1} \rangle) \neq g(\langle a_0, \ldots, a_n \rangle)$$

we let:

$$f_a(x) \simeq \begin{cases} a_x & \text{if } x \leq n \\ \varphi_{g(\langle a_0, \ldots, a_n \rangle)}(x) & \text{if } x > n \text{ and } g \text{ is not forced to change} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

As argued above, we thus obtain an r.e. class of partial recursive functions containing $\mathcal{C}$. It only remains to show that there is a recursive function d such that, if a and b are sequence numbers among the ones considered, then $d(a, b)$ is an upper bound to an argument on which f_a and f_b disagree.

It is enough to let $d(a, b)$ be the maximum of the lengths of a and b. There are two cases to consider:

- If a and b are incompatible, then they differ on some component below $d(a,b)$. But f_a and f_b agree with a and b, respectively, and so they differ on some argument below $d(a,b)$.

- If a and b are compatible, suppose e.g. that a is contained in b, and thus $d(a,b)$ is just the length of b.

  Since both a and b have been considered, g must have changed its guess on some intermediate sequence number, i.e. on some initial segment of b of length $n < d(a,b)$. Then either f_a did not agree with b up to n, and then it differs from f_b for one reason (because by definition f_b does agree with b up to its length), or it did agree, and then it differs from f_b for another reason (because by definition f_a stops being defined at n if not before, while f_b is defined there). $\Box$

We now introduce a variation of 4.3.

Definition 4.5 *An r.e. class* $\{\varphi_{h(e)}\}_{e\in\omega}$ *of partial recursive functions is* **effectively discrete** *if there is a uniform procedure to determine, for every* e *and all almost all* i*, an upper bound to arguments on which* $\varphi_{h(e)}$ *disagrees with* $\varphi_{h(i)}$.

Notice how effective discreteness relates to effective separability: it is weaker because it does not require the existence of a recursive enumeration of the given class without repetitions, and it allows instead for finitely many repetions of each function; and it is stronger because it provides a bound that depends only on e, and not on both e and i.

The next result shows that weakening and strengthening compensate, and that the new notion still characterizes the same classes as the old one.

Theorem 4.6 Second Number-Theoretic Characterization of *EX* (Freivalds, Kinber and Wiehagen [1984]) *A class* $\mathcal{C}$ *of total recursive functions is in* EX *if and only if it is a subclass of an effectively discrete r.e. class of partial recursive functions.*

Proof. Given an effectively discrete r.e. class $\{\varphi_{h(e)}\}_{e\in\omega}$, there is a recursive function d (for 'discreteness') such that, for every e, there are at most finitely many i such that $\varphi_{h(e)}(x) \simeq \varphi_{h(i)}(x)$ for all $x \leq d(e)$. One can define a recursive function g that identifies by explanation any subclass $\mathcal{C}$ of the given class, as follows:

- on the empty list, g takes value 0

- on the list $\langle a_0, \ldots, a_n\rangle$, g takes value 0, unless there are indices e such that:

  - $e \leq n$
  - $d(e) \leq n$

– for all $x \leq d(e)$, $\varphi_{h(e)}(x) \simeq a_x$ in at most n steps

– for all x such that $d(e) < x \leq n$, if $\varphi_{h(e)}(x)$ converges in at most n steps, then $\varphi_{h(e)}(x) \simeq a_x$.

In this case one lets $I_{\langle a_0,\ldots,a_n\rangle}$ be the set of the $h(e)$'s corresponding to all such e's, and defines $g(\langle a_0,\ldots,a_n\rangle)$ as an index of the function that, on any input x, dovetails computations of $\varphi_{h(e)}(x)$ for all $h(e) \in I_{\langle a_0,\ldots,a_n\rangle}$, and outputs the first convergent value.

It is easy to show that $\mathcal{C}$ is identified by explanation via g.

In the opposite direction, let $\mathcal{C}$ be identifiable by explanation via g. By letting $d(a)$ be the length of a, one sees that the r.e. class $\{f_a\}_{a\in\omega}$ defined in the proof of 4.4 is effectively discrete. □

Turning now to complexity-theoretic characterizations, the most natural weakening of 3.5 is obtained by dropping the restriction that the functional H be total on all total recursive functions: this is however too weak, and will be used in 5.3 to characterize EX^*. For a characterization of EX, the following stronger notion is appropriate.

Definition 4.7 *If f is a recursive function and H is a partial recursive functional, then f is* **very H-honest** *if*

$$(\exists e)[f \simeq \varphi_e \ \wedge \ (\forall_\infty x)(H(\varphi_e, x)\downarrow \wedge (\max_{y\leq x} \Phi_e(y)) \leq H(\varphi_e, x))].$$

Thus, while the value $H(\varphi_e, x)$ almost always bounds $\Phi_e(x)$ for H-honest functions, it actually bounds $\max_{y\leq x} \Phi_e(y)$ for very H-honest functions.

Theorem 4.8 Complexity-Theoretic Characterization of EX (Wiehagen and Liepe [1976], Wiehagen [1978]) *A class $\mathcal{C}$ of total recursive functions is in EX if and only if it is a class of very H-honest functions, for some partial recursive functional H.*

Proof. Given a partial recursive functional H, notice that if φ_e is total and very H-honest then there is a constant k such that

$$(\forall x \geq k)[H(\varphi_e, x)\downarrow \Rightarrow (\max_{y\leq x} \Phi_e(y)) \leq H(\varphi_e, x)].$$

One can define a recursive function g that that identifies by explanation the class of all total very H-honest functions, as follows:

- on the empty list, g takes the value 0
- on the list $\langle a_0,\ldots,a_n\rangle$, g takes the value e for the smallest pair $\langle e, k\rangle$ defined as follows, if there is one, and n otherwise:

- $\langle e, k \rangle \leq n$
- for all x such that $k \leq x \leq n$, if $H(\langle a_0, \ldots, a_n \rangle, x)$ converges in at most n steps (where $\langle a_0, \ldots, a_n \rangle$ is considered as the partial function giving value a_x to $x \leq n$ and undefined otherwise) then, for all $y \leq x$,

$$\Phi_e(y) \leq H(\langle a_0, \ldots, a_n \rangle, x) \qquad \text{and} \qquad \varphi_e(y) \simeq a_y.$$

It is easy to show that g identifies by explanation the class of all total very H-honest functions.

In the opposite direction, let $\mathcal{C}$ be identifiable by explanation via g, and define

$$H(\alpha, x) \simeq \max\{\Phi_{g(\langle \alpha(0), \ldots, \alpha(x) \rangle)}(y) : y \leq x\}.$$

H is a partial recursive functional by definition.

Since g identifies $\mathcal{C}$, if $f \in \mathcal{C}$ then g has a limit e on f, and $\varphi_e \simeq f$. If x_0 is a point after which g does not change anymore on f then, for all $x \geq x_0$,

$$H(f, x) \simeq \max\{\Phi_{g(\langle f(0), \ldots, f(x) \rangle)}(y) : y \leq x\} = \max_{y \leq x} \Phi_e(y).$$

Then e is a witness of the fact that f is very H-honest. $\Box$

5 Identification by explanation with finite errors

There are two opposite ways of relaxing the requirements imposed in the definition of EX: we can be less restrictive on the function we identify in the limit, and ask not for the real f, but only for a finite approximation to it; or we can be less restrictive on the convergence of our guesses. We investigate here the first option, and in next sections the second one.

Finitely many exceptions to the range of an explanation are readily accepted in science: for example, Newtonian mechanics was accepted even if it did not account correctly for the motion of Mercury's perielion. The next notion is thus not without interest.

Definition 5.1 (Blum and Blum [1975]) *A class $\mathcal{C}$ of total recursive functions is* **identifiable by explanation with finitely many errors** *($\mathcal{C} \in \boldsymbol{EX}^*$) if there is a total recursive function g such that, for every $f \in \mathcal{C}$:*

- $\lim_{n \to \infty} g(\langle f(0), \ldots, f(n) \rangle)$ *exists*
- $\varphi_{\lim_{n \to \infty} g(\langle f(0), \ldots, f(n) \rangle)} \simeq^* f$.

In other words, $g(\langle f(0), \ldots, f(n) \rangle)$ provides a guess to an index of f, and the guess stabilizes (from a certain point on) on an index of a finite variant of f.

Notice how, since we are now interested in guessing not the real function f but only a finite variant of it, we cannot request that the current guess agrees with the available information, and thus no analogue of EX_{cons} makes sense in this context.

Also, while in the definitions of EX_{cons} and EX the temporary guesses could be programs for partial functions but the final guesses had to be programs for a total function, here *all* guesses may actually be programs for partial functions (although the final guess must compute a function that can be undefined only on finitely many arguments, since it has to be a finite variant of a total functions).

Proposition 5.2 (Blum and Blum [1975]) *$EX \subset EX^*$.*

Proof. The inclusion is obvious by definition. To show that it is proper, we want to find a class $\mathcal{C}$ of total recursive functions which is in EX^* but not in EX. Let

$$\mathcal{C} = \{f : f \simeq^* \varphi_{f(0)}\},$$

i.e. the class of total recursive functions such that $f(0)$ is an index of a finite variant of f.

$\mathcal{C} \in EX^*$, by letting $g(\langle\ \rangle) = 0$ and $g(\langle a_0, \ldots, a_n\rangle) = a_0$. In other words, every function in $\mathcal{C}$ gives away a program for a finite variant of itself as its first value, thus making identification by explanation with finite errors trivial.

To show that $\mathcal{C} \notin EX$, suppose g is a recursive function such that

$$f \simeq^* \varphi_{f(0)} \;\Rightarrow\; f = \varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)}.$$

The idea is to construct $f \in \mathcal{C}$ such that either $\lim_{n\to\infty} g(\langle f(0), \ldots, f(n)\rangle)$ does not exist (because g changes its guess infinitely often), or it exists but it is not an index of f (being an index of a partial function).

There is no problem in forcing $f(0)$ to be an index of the function f to be defined, using the Fixed-Point Theorem as in the proof of 4.2. We thus concentrate on the definition of the remaining values of f. At each stage $n+1$ we have already defined all values of f up to n with one exception a_n, which is used to satisfy the idea above. There are three possible cases:

1. If $g(\langle f(0), \ldots, f(a_n - 1)\rangle) \neq g(\langle f(0), \ldots, f(a_n - 1), 0, f(a_n + 1), \ldots, f(n)\rangle)$ then we define $f(a_n) = 0$, thus ensuring that g has changed its value once, and let $a_{n+1} = n + 1$ (since now every value up to n has been defined).

2. If case 1 does not hold, and $\varphi_{g(\langle f(0),\ldots,f(a_n-1)\rangle)}(a_n)$ converges in at most n steps, then we define $f(a_n) = 1 - \varphi_{g(\langle f(0),\ldots,f(a_n-1)\rangle)}(a_n)$, thus ensuring that f is different from the function guessed on the initial segment of f of length a_n, and let $a_{n+1} = n + 1$ as above.

3. If cases 1 and 2 do not hold, we let $f(n+1) = 0$, thus defining f on one more value, and $a_{n+1} = a_n$ (to have a new shot at it at the next stage).

At the end, there are two possible cases:

- $\lim_{n\to\infty} a_n$ *does not exist*
  This means that we move a_n infinitely often, in particular the function f is total. Moreover, $f = \varphi_{f(0)}$ by construction, in particular $f \in \mathcal{C}$. By hypothesis then $\lim_{n\to\infty} g(\langle f(0), \ldots, f(n)\rangle)$ should exist, and f should be $\varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)}$.

  But if $\lim_{n\to\infty} g(\langle f(0), \ldots, f(n)\rangle)$ exists, then case 1 holds only finitely many times. Since a_n moves infinitely often, after a certain stage it must do so because of case 2. But then f is different from $\varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)}$ (because it disagrees with infinitely many guesses, and hence it must disagree with the final guess).

- $\lim_{n\to\infty} a_n$ *exists*
  If a is this limit, then f is defined everywhere except in a, since otherwise a would have moved as soon as $f(a)$ had been defined. Let f_1 be the extension of f obtained by letting $f_1(a) = 0$, and $f_1(x) = f(x)$ if $x \neq a$.

  Then $g(\langle f_1(0), \ldots, f_1(a-1)\rangle) = \lim_{n\to\infty} g(\langle f_1(0), \ldots, f_1(n)\rangle)$, otherwise case 1 would have taken place, and a would have moved.

  Moreover, $\varphi_{\lim_{n\to\infty} g(\langle f_1(0),\ldots,f_1(n)\rangle)}(a)$ must be undefined, otherwise case 2 would have taken place, and a would have moved.

  $f_1(0) = f(0)$ is by definition an index of f, and hence of a finite variant of f_1. So $f_1 \simeq^* \varphi_{f_1(0)}$, but $f_1 \not\simeq \varphi_{\lim_{n\to\infty} g(\langle f_1(0),\ldots,f_1(n)\rangle)}$, because the former is defined on a but the latter is not. $\square$

The proof of the previous result shows that actually

$$\mathcal{C} = \{f : f \simeq^* \varphi_{f(0)}, \text{ with at most one disagreement point}\}$$

is in $EX^* - EX$. In particular, even allowing for a *single* exception in the explanation of a class of phenomena already gives more power (in the sense of being able to explain more classes of phenomena) than requiring an explanation to be always correct.

Moreover, the proof also shows how the exception might occur in a place in which the explanation does not give any answer, being undefined. Since the Halting Problem is unsolvable, such an explanation (being correct when it does provide an answer, and incorrect in a divergent point) cannot be refuted (in a recursive way). In other words, *there are incomplete and irrefutable explanations*, and they do not satisfy a form of Popper Refutability Principle (Popper [1934]), according to which incorrect scientific explanations should be refutable.

We do not know of any number-theoretic characterization of EX^*. For a complexity-theoretic characterization, we already have at hand the appropriate notion.

Theorem 5.3 Complexity-Theoretic Characterization of $\boldsymbol{EX^*}$ (Wiehagen [1978], Kinber) *A class $\mathcal{C}$ of total recursive functions is in EX^* if and only if it is a class of H-honest functions, for some partial recursive functional H and w.r.t. space complexity measure.*

Proof. Given a partial recursive functional H, notice that if φ_e is H-honest then there is a constant k such that

$$(\forall x \geq k)[H(\varphi_e, x)\downarrow \Rightarrow \Phi_e(x) \leq H(\varphi_e, x)].$$

One can define a recursive function g that identifies by explanation with finite errors the class of all total H-honest functions, as follows:

- on the empty list, g takes the value 0
- on the list $\langle a_0, \ldots, a_n \rangle$, g takes the value e for the smallest pair $\langle e, k \rangle$ defined as follows, if there is one, and n otherwise:
  - $\langle e, k \rangle \leq n$
  - for all x such that $k \leq x \leq n$, if $H(\langle a_0, \ldots, a_n \rangle, x)$ converges in at most n steps (where $\langle a_0, \ldots, a_n \rangle$ is considered as the partial function giving value a_x to $x \leq n$ and undefined otherwise) then

$$\Phi_e(x) \leq H(\langle a_0, \ldots, a_n \rangle, x) \qquad \text{and} \qquad \varphi_e(x) \simeq a_x.$$

It is easy to show that g identifies by explanation with finite errors the class of all total H-honest functions.

In the opposite direction, let $\mathcal{C}$ be identifiable by explanation with finite errors via g, and define

$$H(\alpha, x) \simeq \Phi_{g(\langle \alpha(0), \ldots, \alpha(x) \rangle)}(x).$$

H is a partial recursive functional by definition.

Since g identifies $\mathcal{C}$ with finite errors, if $f \in \mathcal{C}$ then g has a limit e on f, and $\varphi_e \simeq^* f$. If x_0 is a point after which g does not change anymore on f then, for all $x \geq x_0$,

$$H(f, x) \simeq \Phi_{g(\langle f(0), \ldots, f(x) \rangle)}(x) \simeq \Phi_e(x).$$

If moreover $x_1 \geq x_0$ is a point after which φ_e and f agree, then $\varphi_e(x)$ converges for all $x \geq x_1$, and hence

$$(\forall_\infty x)[H(f, x)\downarrow \wedge \Phi_e(x) \leq H(f, x)].$$

We cannot claim that f is H-honest yet, since e is not an index of f (as required by Definition 3.4), but only of a finite variant of it. However, if we consider a complexity measure such as space, whose complexity classes are closed under finite variants, the result then follows. □

6 Behaviorally correct (consistent) identification

In the definitions of EX_{cons} and EX we required the final explanation of each phenomenon in a given class to be *intensionally* unique. The next definition only asks for *extensional* uniqueness, and allows for the possibility of not having a final intensional explanation (i.e. it allows infinitely many changes in the program, although still only finitely many in the function defined by it).

Definition 6.1 (Feldman [1972], Barzdin [1974]) *A class $\mathcal{C}$ of total recursive functions is* **behaviorally correctly and consistently identifiable** *($\mathcal{C} \in \boldsymbol{BC_{cons}}$) if there is a total recursive function g such that, for every sequence number $\langle a_0, \ldots, a_n \rangle$:*

- $\varphi_{g(\langle a_0,\ldots,a_n\rangle)}(x) \simeq a_x$ *for all $x \leq n$,*

and for every $f \in \mathcal{C}$ and almost every n,

- $\varphi_{g(\langle f(0),\ldots,f(n)\rangle)} = f$.

In other words, $g(\langle f(0), \ldots, f(n)\rangle)$ provides a guess to an index of f consistent with the available information, and the guess stabilizes (from a certain point on) on indices of f.

$\mathcal{C}$ is **behaviorally correctly identifiable** *($\mathcal{C} \in \boldsymbol{BC}$) if the first condition on g is dropped.*

First of all we see that the new notions of inference just introduced coincide, so that no analogue of 4.2 holds.

Proposition 6.2 *$BC_{cons} = BC$.*

Proof. $BC_{cons} \subseteq BC$ by definition. Conversely, if a procedure g outputs the guesses for functions in a class $\mathcal{C} \in BC$, then these guesses on $f \in \mathcal{C}$ are correct, and hence consistent in the limit. One can modify such a procedure, and output at stage n an index of the function that agrees with f on the arguments $\leq n$, and with $\varphi_{g(\langle f(0),\ldots,f(n)\rangle)}$ otherwise. I.e. one outputs $g_1(\langle f(0), \ldots, f(n)\rangle)$, for any g_1 such that

$$\varphi_{g_1(\langle a_0,\ldots,a_n\rangle)}(x) \simeq \begin{cases} a_x & \text{if } x \leq n \\ \varphi_{g(\langle a_0,\ldots,a_n\rangle)}(x) & \text{otherwise.} \end{cases}$$

Since if $f \in \mathcal{C}$ then $g(\langle f(0), \ldots, f(n)\rangle)$ stabilizes on indices of f, g_1 still identifies $\mathcal{C}$ in a behaviorally correct way, and is consistent by definition. Thus $\mathcal{C}$ is in BC_{cons}. □

We now show that the new notion of inference is related to (and weaker than) the ones studied so far.

Proposition 6.3 (Steel) $EX^* \subseteq BC$.

Proof. If a procedure g outputs the guesses for functions in a class $\mathcal{C} \in EX^*$, then these guesses on $f \in \mathcal{C}$ are correct in the limit, but only for finite variants of f. One can modifiy such a procedure as in 6.2, by considering g_1 such that

$$\varphi_{g_1(\langle a_0,\ldots,a_n\rangle)}(x) \simeq \begin{cases} a_x & \text{if } x \leq n \\ \varphi_{g(\langle a_0,\ldots,a_n\rangle)}(x) & \text{otherwise.} \end{cases}$$

Since if $f \in \mathcal{C}$ then $g(\langle f(0),\ldots,f(n)\rangle)$ stabilizes on an index of a finite variant of f, after a finite number of stages all modifications will compute the same function, and will be correct on all values. $\Box$

Notice that in the previous proof we cannot conclude $\mathcal{C} \in EX$, because if $f \in \mathcal{C}$ then $g_1(\langle f(0),\ldots,f(n)\rangle)$ codes a different program for every n, although a program obtained by patching up an eventually fixed program (for a finite variant of f) on an increasing number of arguments.

Proposition 6.4 (Barzdin [1974], Case and Smith [1983], Harrington) $BC - EX^* \neq \emptyset$.

Proof. We want to find a class of $\mathcal{C}$ of total recursive functions which is in BC but not in EX^*. Let

$$\mathcal{C} = \{f : (\forall_\infty x)(f = \varphi_{f(x)})\},$$

i.e. the class of total recursive functions such that $f(x)$ is an index of f, for almost every x.

$\mathcal{C} \in BC$, by letting $g(\langle\ \rangle) = 0$ and $g(\langle a_0,\ldots,a_n\rangle) = a_n$. In other words, every function in $\mathcal{C}$ almost always gives away programs for itself as values, thus making behaviorally correct identification trivial.

To show that $\mathcal{C} \notin EX^*$, suppose g is a recursive function such that

$$(\forall_\infty x)(f = \varphi_{f(x)}) \ \Rightarrow\ f \simeq^* \varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)}.$$

The idea is to construct $f \in \mathcal{C}$ such that either $\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)$ does not exist (because g changes its guess infinitely often), or it exists but it is not an index of f (being an index of a function which is not a finite variant of f).

We construct f by initial segments σ_s, starting from $\sigma_0 = \emptyset$, and let n_s be the greatest argument on which σ_s is defined. A natural strategy to satisfy the condition

$$f \not\simeq^* \varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)},$$

if the limit exists, is to make

$$f(x) \not\simeq \varphi_{g(\langle\sigma_s(0),\ldots,\sigma_s(n_s)\rangle)}(x),$$

for a new x at every stage.

At stage $s+1$ we thus wait until $\varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}(x)$ converges for some $x > n_s$, and then diagonalize. But since at the same time we want $f \in \mathcal{C}$, we will have to give f its own indices as values; we then choose two distinct indices e_0 and e_1 of f (by the Fixed-Point Theorem), and when $\varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}(x)$ converges for some $x > n_s$ we extend σ_s by letting $\sigma_{s+1}(x) = e_i$, for an i (equal to 0 or 1) such that

$$e_i \neq \varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}(x).$$

To have an initial segment, we also give all the arguments less than x and not in the domain of σ_s one of the values e_0 or e_1 (say, e_0).

Obviously, there is no guarantee that $\varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}(x)$ will ever converge, for any $x > n_s$; and if it does not, then the construction would stall. We thus need an alternative strategy, and at stage $s+1$ we also build an additional function f_s, that will work if the construction of f stalls at that step.

Again, to ensure $f_s \in \mathcal{C}$ we will have to give f_s its own indices as values; we thus choose an index i_s of f_s (by the Fixed-Point Theorem), and at every step in the dovetailed computations of $\varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}(x)$ for $x > n_s$ we define f_s as i_s on a new argument, with the intent that if $\varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}(x)$ does not converge for any $x > n_s$, then f_s will be equal to i_s from a certain point on, and then automatically in $\mathcal{C}$.

This would not be of great help, unless we also ensured that

$$f_s \not\simeq^* \varphi_{\lim_{n\to\infty} g(\langle f_s(0),\dots,f_s(n)\rangle)}.$$

We are working under the hypothesis that $\varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}(x)$ does not converge for any $x > n_s$, and hence that $\varphi_{g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle)}$ is a finite function. It is thus natural to use this, by letting f_s extend σ_s; thus we will define f_s as equal to i_s only for $x > n_s$, and equal to $\sigma_s(x)$ if $x \leq n_s$. If

$$g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle) = \lim_{n\to\infty} g(\langle f_s(0),\dots,f_s(n)\rangle),$$

then

$$f_s \not\simeq^* \varphi_{\lim_{n\to\infty} g(\langle f_s(0),\dots,f_s(n)\rangle)}$$

because f_s is total, while the right hand side is a finite function.

Obviously, there is no guarantee that

$$g(\langle\sigma_s(0),\dots,\sigma_s(n_s)\rangle) = \lim_{n\to\infty} g(\langle f_s(0),\dots,f_s(n)\rangle).$$

But if this does not hold, then g changes value on f_s sooner or later, and we can take advantage of this to go back to the definition of f. Indeed, by the first strategy we tried to make $\lim_{n\to\infty} g(\langle f(0),\dots,f(n)\rangle)$ not an index of f, while with the present back up strategy we are trying to ensure that the limit does not exist.

In other words, we continue to define f_s as i_s for more and more arguments, until we discover that for an initial segment τ of f_s,

$$g(\tau) \neq g(\langle \sigma_s(0), \ldots, \sigma_s(n_s) \rangle).$$

If this happens, then we let $\sigma_{s+1} = \tau$. This is consistent with what was previously done, i.e. $\sigma_{s+1} \supseteq \sigma_s$, because f_s extends σ_s.

Notice that defining $\sigma_{s+1} = \tau$ gives f value i_s for a number of arguments; to avoid ruining the strategy for $f \in \mathcal{C}$, we better make i_s an index of f. Thus, we stop defining new values of f_s as i_s (since, in any case, f_s has lost its role as possible witness of the fact that g does not identify $\mathcal{C}$), and from now on the definition of f_s will just copy the definition of f.

We now have to argue that the proposed construction works. There are two possible cases:

- *all stages terminate*

  Each value of f is either one of e_0 and e_1, or some i_s; hence an index of f, either by the initial choice of e_0 and e_1, or by construction of f_s. In particular, $(\forall x)(f = \varphi_{f(x)})$, and $f \in \mathcal{C}$.

  If g has no limit on f, there is nothing to prove. If g does have a limit on f, then we look at the construction of f. Every time the second part of the construction is applied at some stage, g changes value on f; since g has a limit on f by hypothesis, the second part of the construction can thus by applied only finitely many times. Then the first part of the construction is applied almost always; but every time it is applied, a disagreement between f and the current guess is enforced, and once the guess has stabilized, a disagreement with the final guess is enforced. In particular, $f \not\simeq^* \varphi_{\lim_{n\to\infty} g(\langle f(0),\ldots,f(n)\rangle)}$.

- *some stage $s+1$ does not terminate*

  By construction, f_s is then total. Moreover, almost all of its values are i_s, hence an index of f_s by the choice of i_s. In particular, $(\forall_\infty x)(f_s = \varphi_{f_s(x)})$, and $f_s \in \mathcal{C}$.

  By construction

  $$g(\langle \sigma_s(0), \ldots, \sigma_s(n_s) \rangle) = \lim_{n\to\infty} g(\langle f_s(0), \ldots, f_s(n) \rangle),$$

  otherwise stage $s+1$ would terminate by the second part of the construction. Moreover, $\varphi_{g(\langle \sigma_s(0),\ldots,\sigma_s(n_s)\rangle)}$ is a finite function, because if it were defined on some $x > n_s$ then stage $s+1$ would terminate by the first part of the construction. In particular, $f_s \not\simeq^* \varphi_{\lim_{n\to\infty} g(\langle f_s(0),\ldots,f_s(n)\rangle)}$, because f_s is total.

We have made infinitely many appeals to the Fixed-Point Theorem in the constructions of f and of the f_s (which is not surprising, since f had to be self-referential infinitely often, to be in $\mathcal{C}$). There would be no problem if these constructions were independent, but they are instead related one to the others (f may use the index i_s, and f_s may mimic f). We thus still have to make sure that it is really possible to construct an infinite sequence of functions, simultaneously using indices for all of them.

Since a sequence of indices can be thought of as the range of a function f, and the previous construction actually produces a recursive functional F, the next result provides a formal justification.

- **Functional Recursion Theorem (Case [1974])** *Given a partial recursive functional F, there is a total recursive function f such that*

$$(\forall e)(\forall x)[\varphi_{f(e)}(x) \simeq F(f, e, x)].$$

Proof. It is enough to find a recursive function g such that

$$\varphi_{\varphi_{g(i)}(e)}(x) \simeq F(\varphi_i, e, x).$$

Then the usual Fixed-Point Theorem gives an i such that $\varphi_i \simeq \varphi_{g(i)}$, and

$$\varphi_{\varphi_i(e)}(x) \simeq \varphi_{\varphi_{g(i)}(e)}(x) \simeq F(\varphi_i, e, x).$$

By taking $f \simeq \varphi_i$, we obtain the result.

To get g, consider the partial recursive function

$$\psi(i, e, x) \simeq F(\varphi_i, e, x).$$

By the S^m_n-Theorem, there is a recursive function h such that

$$\varphi_{h(i,e)}(x) \simeq \psi(i, e, x).$$

By the S^m_n-Theorem again, there is a recursive function g such that

$$\varphi_{g(i)}(e) \simeq h(i, e).$$

Then

$$\varphi_{\varphi_{g(i)}(e)}(x) \simeq \varphi_{h(i,e)}(x) \simeq \psi(i, e, x) \simeq F(\varphi_i, e, x).$$

Notice that h is total because defined by the S^m_n-Theorem, and hence so are $\varphi_{g(i)}$ and φ_i, i.e. f. $\square$

We do not know of any complexity-theoretic characterization of BC. The next result provides a number-theoretic one in the style of the characterization 4.6 of EX.

Definition 6.5 *Given an r.e. class $\{\varphi_{h(e)}\}_{e\in\omega}$ of partial recursive functions, a subclass $\mathcal{C}$ of total recursive functions is called* **effectively weakly discrete** *in it if there is a uniform procedure to determine, for every $f \in \mathcal{C}$ and almost every i, an upper bound to arguments on which f disagrees with $\varphi_{h(i)}$, if they disagree at all.*

The notion just introduced is a double weakening of effective discreteness: first of all, it allows for infinitely many repetitions in the given class, while only finitely many were allowed in 4.5; secondly, it is not a global condition on the given class, but rather a local one on the given subclass.

Theorem 6.6 Characterization of *BC* (Wiehagen) *A class $\mathcal{C}$ of total recursive functions is in BC if and only if it is an effectively weakly discrete subclass of an r.e. class of partial recursive functions.*

Proof. Given a class $\mathcal{C}$ effectively weakly discrete in an r.e. class $\{\varphi_{h(e)}\}_{e\in\omega}$, there is a recursive function d such that, for every $\varphi_{h(e)} \in \mathcal{C}$, there are at most finitely many i such that, if $\varphi_{h(e)}(x) \simeq \varphi_{h(i)}(x)$ for all $x \leq d(e)$, then $\varphi_{h(e)}$ and $\varphi_{h(i)}$ are different. One can define a recursive function g that behaviorally correctly identifies $\mathcal{C}$ as in the proof of 4.6.

In the opposite direction, let $\mathcal{C}$ be behaviorally correctly identifiable via g. Then every function in $\mathcal{C}$ is of the following form, for some sequence number a (coding a list $\langle a_0, \ldots, a_n\rangle$ for some n):

$$f_a(x) \simeq \begin{cases} a_x & \text{if } x \leq n \\ \varphi_{g(\langle a_0,\ldots,a_n\rangle)}(x) & \text{otherwise} \end{cases}$$

(more precisely, $f \in \mathcal{C}$ is equal to f_a for any sequence number a coding the first n values of f, for any n greater than the point after which g only outputs indices of f). Any such f_a is partial recursive uniformly in a, and by the S^m_n-Theorem there is then a recursive function h such that $\varphi_{h(a)} = f_a$. Thus the class $\{f_a\}_{a\in\omega}$ is an r.e. class of partial recursive functions containing $\mathcal{C}$.

Let $d(a)$ be the length of a. Suppose $f \in \mathcal{C}$, and f and f_a agree up to $d(a)$, i.e. a is an initial segment of f. There are two cases: either $d(a)$ is greater than the point after which g only outputs indices of f, and then $f_a = f$ by definition; or $d(a)$ is not greater than such a point, and this can happen only for finitely many a's. □

7 Behaviorally correct identification with finite errors

We next relax BC in a way similar to what we did for EX in 5.1.

Definition 7.1 (Osherson and Weinstein [1982], Case and Smith [1983]) *A class $\mathcal{C}$ of total recursive functions is* **behaviorally correctly identifiable with finitely many errors** *($\mathcal{C} \in \boldsymbol{BC^*}$) if there is a recursive function g such that, for every $f \in \mathcal{C}$ and almost every n,*

$$\varphi_{g(\langle f(0),\ldots,f(n)\rangle)} \simeq^* f.$$

In other words, $g(\langle f(0), \ldots, f(n)\rangle)$ provides a guess to an index of f, and the guess stabilizes (from a certain point on) on indices of finite variants of f.

Notice that BC^* is a very weak notion: not only the guesses can change infinitely often, as they did for BC, but even the functions that they compute can do so. The only requirement is that each such function eventually be a finite variant of f.

The next result shows that we have finally reached the possible limits of inductive inference.

Theorem 7.2 Characterization of $\boldsymbol{BC^*}$ (Harrington) *The class of all (and hence any class of) total recursive functions is in BC^*.*

Proof. Let $\mathcal{C}$ be the class of all total recursive functions. To show that $\mathcal{C} \in BC^*$, the idea is to take advantage of the fact that if f is a total recursive function and e is its least index, then

$$(\forall i < e)(\exists x)[\varphi_i(x) \not\simeq f(x)],$$

i.e. for every smaller index there is a disagreement with f. Notice that, however, φ_i might disagree with f on x either because it converges to a different value, or because it does not converge (while f does, being total).

Let $g(\langle\ \rangle) = 0$. Given $a_0, \ldots, a_n$, we let $g(\langle a_0, \ldots, a_n\rangle)$ be the program of the total recursive function defined as follows. On input x, see if there is e_x such that:

- $e_x \leq n$
- for every $m \leq n$, $\varphi_{e_x}(m)$ converges in at most x steps to a_m.

Then

$$\varphi_{g(\langle a_0,\ldots,a_n\rangle)}(x) \simeq \begin{cases} \varphi_{e_x}(x) & \text{for the least such } e_x\text{, if one exists} \\ 0 & \text{otherwise.} \end{cases}$$

Let now f be a total recursive function. We show that, for any sufficiently big n, $g(\langle f(0), \ldots, f(n)\rangle)$ is an index of a finite variant of f. Indeed, let e be the least index of f, and n_0 be big enough so that:

- $e \leq n_0$

- for every $i < e$ there is an $m \leq n_0$ such that $\varphi_i(m) \not\simeq f(m)$.

For any $n \geq n_0$, let x_n be big enough so that:

- for every $m \leq n$, $\varphi_e(m)$ converges in at most x_n steps.

Then, for every $x \geq x_n$,

$$\varphi_{g(\langle f(0),\ldots,f(n)\rangle)}(x) \simeq \varphi_e(x) \simeq f(x),$$

and thus $g(\langle f(0), \ldots, f(n)\rangle)$ is an index of a finite variant of f. □

Proposition 7.3 (Bardzin [1974], Case and Smith [1983]) *$BC \subset BC^*$.*

Proof. The inclusion is obvious by definition. To show that it is proper, it is enough to show that the class of total recursive functions is not in BC. Let g be a recursive function such that

$$f \text{ total recursive } \Rightarrow \lim_{n\to\infty} \varphi_{g(\langle f(0),\ldots,f(n)\rangle)} = f.$$

We define a total recursive function f such that $\lim_{n\to\infty} \varphi_{g(\langle f(0),\ldots,f(n)\rangle)}$ does not exist, contradicting the hypothesis on g. The idea is that f will repeat the same value a number of times sufficient to make g converge, and then will diagonalize against the program thus obtained: since we do this infinitely often, then the function defined by g changes infinitely often on f, and thus it has no limit.

Start with $f(0) = 0$. The function f_0 identically equal to 0 is recursive and, by hypothesis on g, $\lim_{n\to\infty} \varphi_{g(\langle f_0(0),\ldots,f_0(n)\rangle)}$ exists. Thus there must exist n and $m > n$ such that $\varphi_{g(\langle f_0(0),\ldots,f_0(n)\rangle)}(m)\downarrow$, and we can effectively find them (by dovetailing computations). Let

$$f(x) = \begin{cases} f_0(x) & \text{if } x \leq n \\ 0 & \text{if } n \leq x < m \\ \varphi_{g(\langle f_0(0),\ldots,f_0(n)\rangle)}(x) + 1 & \text{if } x = m, \end{cases}$$

and iterate the procedure (by considering, at each step s, the function f_s extending the values already obtained by a sequence of 0's). □

In conclusion, the sequence of results proved in 2.2, 2.3, 3.2, 3.3, 4.2, 5.2, 6.2, 6.3, 6.4, and 7.3 provide the following picture of inductive inference classes:

$$NV \subset EX_{cons} \subset EX_{rel} \subset EX \subset EX^* \subset BC_{cons} = BC \subset BC^*.$$

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